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$\boxed{10}$

(1) $\int_0^4 \frac{1}{\sqrt{4-x}}dx$

$= \lim_{\varepsilon\to+0}\int_0^{4-\varepsilon}\frac{1}{\sqrt{4-x}}dx$

$= \lim_{\varepsilon\to+0}\left[-2\sqrt{4-x}\right]_0^{4-\varepsilon}$

$= \lim_{\varepsilon\to+0}\left\{-2\sqrt{\varepsilon}+2\sqrt{4}\right\} = 4$

(2) $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx$

$= \lim_{\varepsilon,\varepsilon'\to+0}\int_{-1+\varepsilon}^{1-\varepsilon'}\frac{1}{\sqrt{1-x^2}}dx$

$= \lim_{\varepsilon,\varepsilon'\to+0}\left[\sin^{-1}x\right]_{-1+\varepsilon}^{1-\varepsilon'}$

$= \lim_{\varepsilon,\varepsilon'\to+0}\left\{\sin^{-1}(1-\varepsilon')-\sin^{-1}(-1+\varepsilon)\right\}$

$= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi$

(3) $\lim_{h\to+0}\int_h^1\frac{1}{\sqrt{x+x^2}}dx$

$= \lim_{h\to+0}\int_h^1\frac{1}{\sqrt{(x+\frac{1}{2})^2-(\frac{1}{2})^2}}$

$= \lim_{h\to+0}\left[\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|\right]_h^1$

$= \log\left(1+\frac{1}{2}+\sqrt{2}\right)-\log\left(\frac{1}{2}\right) = \log\left(3+2\sqrt{2}\right)$

(4) $\int_0^1\frac{x^n}{\sqrt{1-x^2}}dx = \lim_{\varepsilon\to+0}\int_0^{1-\varepsilon}\frac{x^n}{\sqrt{1-x^2}}dx$

$x=\sin\theta$とおくと $dx=\cos\theta\, d\theta$

x	$0\longrightarrow 1$
θ	$0\longrightarrow\frac{\pi}{2}$

$(与式) = \lim_{\varepsilon\to+0}\int_0^{\sin^{-1}(1-\varepsilon)}\frac{\sin^n\theta}{\cos\theta}\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}}\sin^n\theta\, d\theta \quad = I_n$ とおくと

$I_n = \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos^2\theta\,(n-1)\sin^{n-2}\theta\, d\theta$

$\therefore I_n = \frac{n-1}{n}I_{n-2}$

ゆえに $I_{n+2} = \frac{n+1}{n+2}I_n$

(i) $n=2k$のとき

$I_n = \frac{1\cdot3\cdot\cdots\cdot(2k-1)}{2\cdot4\cdot\cdots\cdot2k}\cdot\frac{\pi}{2}$

(ii) $n=2k+1$のとき

$I_n = \frac{2\cdot4\cdots2k}{1\cdot3\cdots(2k+1)}$

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(5) $\int_{-2}^{\infty} \frac{1}{x^2+4}\,dx$ $\qquad x = 2\tan\theta$ とおく

$$\frac{dx}{d\theta} = \frac{2}{\cos^2\theta}$$

x	$-2 \longrightarrow N$
θ	$-\frac{\pi}{4} \longrightarrow \tan^{-1}\frac{N}{2}$

$$(5式) = \lim_{N\to\infty} \int_{-\frac{\pi}{4}}^{\tan^{-1}\frac{N}{2}} \frac{1}{4(\tan^2\theta+1)} \cdot \frac{2}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2}\left[\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{3}{8}\pi$$

(6) $\int_0^{\infty} x e^{-\frac{x^2}{2}}\,dx$

ここで $\left(-e^{-\frac{x^2}{2}}\right)' = xe^{-\frac{x^2}{2}}$ より

$\lim_{n\to\infty} \int_0^n \left(-e^{-\frac{x^2}{2}}\right)' dx$

$\int_0^n \left(-e^{-\frac{x^2}{2}}\right)' dx$

$= \left[-e^{-\frac{x^2}{2}}\right]_0^n = -e^{-\frac{n^2}{2}} + 1$ より

$\lim_{n\to\infty}\left(e^{-\frac{n^2}{2}} + 1\right) = 1$

(7) $\cos\left(x + \frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}}(\cos x - \sin x)$

よって(5式) $= \int_0^{\infty} e^{-x} \cdot \frac{1}{\sqrt{2}}(\cos x - \sin x)\,dx$

$= \frac{1}{\sqrt{2}}\int_0^{\infty}(e^{-x}\cos x)\,dx - \frac{1}{\sqrt{2}}\int_0^{\infty} e^{-x}\sin x\,dx$ ‥‥ ①

$\int_0^{\infty} e^{-x}\cos x\,dx = A$, $\int_0^{\infty} e^{-x}\sin x = B$ とおく

$B = \lim_{n\to\infty}\left[-e^{-x}\sin x\right]_0^n + \int_0^{\infty} e^{-x}\cos x\,dx = \lim_{n\to\infty} -e^{-n}\sin n + 0 + A$

$= A$

①に $A - B$ を代入して

$(5式) = \frac{1}{\sqrt{2}}(A - B)$

$= \frac{1}{\sqrt{2}}(A - A) = 0$

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(8) $\int_0^1 \sqrt{\frac{x}{1-x}}\,dx = \int_0^1 \frac{\sqrt{x}}{\sqrt{1-\sqrt{x}^2}}\,dx$

$\sqrt{x} = \sin t$ とおくと

x	$0 \to 1-\varepsilon$
t	$0 \to \sin^{-1}\sqrt{1-\varepsilon}$

$dx = 2\sin t \cos t\, dt$

$\therefore \lim_{\varepsilon\to 0}\int_{t=0}^{t=\sin^{-1}\sqrt{1-\varepsilon}} \frac{2\sin^2 t\cos t}{\sqrt{1-\sin^2 t}}\,dt$

$= \int_0^{\frac{\pi}{2}} 2\sin^2 t\,dt$

$= \int_0^{\frac{\pi}{2}} (1-\cos 2t)\,dt$

$= \left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

(9) $\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{a^2+b^2\tan^2 x} \times \frac{1}{\cos^2 x}\,dx$

$x = \mathrm{Tan}^{-1} t$ とおくと

$t = \tan x \quad dt = \frac{1}{\cos^2 x}\,dx$

x	$0 \to \frac{\pi}{2}$
t	$0 \to N$

$= \lim_{N\to\infty}\int_0^N \frac{1}{a^2+b^2t^2}\,dt$

$= \lim_{N\to\infty}\frac{1}{b^2}\int_0^N \frac{1}{(\frac{a}{b})^2+t^2}\,dt$

ここで $t = \frac{a}{b}\tan\theta$ とおくと

$dt = \frac{a}{b}\cdot\frac{1}{\cos^2\theta}\,d\theta$

t	$0 \to \infty$
θ	$0 \to \frac{\pi}{2}$

$= \frac{1}{b^2}\int_0^{\frac{\pi}{2}} \frac{\frac{a}{b}\cdot\frac{1}{\cos^2\theta}}{(\frac{a}{b})^2+(\frac{a}{b}\tan\theta)^2}\,d\theta$

$= \frac{1}{ab}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{ab}[\theta]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2ab}$

(10) $\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}}\,dx \quad (a<b)$

$= \int_a^b \frac{1}{\sqrt{-x^2+(a+b)x-ab}}\,dx$

$= \int_a^b \frac{1}{\sqrt{-(x-\frac{a+b}{2})^2+(\frac{a+b}{2})^2-ab}}$

$= \int_a^b \frac{1}{\sqrt{-(x-\frac{a+b}{2})^2+(\frac{a-b}{2})^2}}$

$x-\frac{a+b}{2} = t$ とおくと $dx = dt$

x	$a \to b$
t	$\frac{a-b}{2}+\varepsilon \to \frac{b-a}{2}-\varepsilon'$

$(与式) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0}}\int_{\frac{a-b}{2}+\varepsilon'}^{\frac{b-a}{2}-\varepsilon'} \frac{1}{\sqrt{(\frac{a-b}{2})^2-t^2}}\,dt$

$= \left[\sin^{-1}\frac{2}{a-b}t\right]_{\frac{a-b}{2}+\varepsilon}^{\frac{b-a}{2}-\varepsilon'}$

$= \sin^{-1}(-1) - \sin^{-1}(1) = \frac{3}{2}\pi - \frac{1}{2}\pi = \pi$