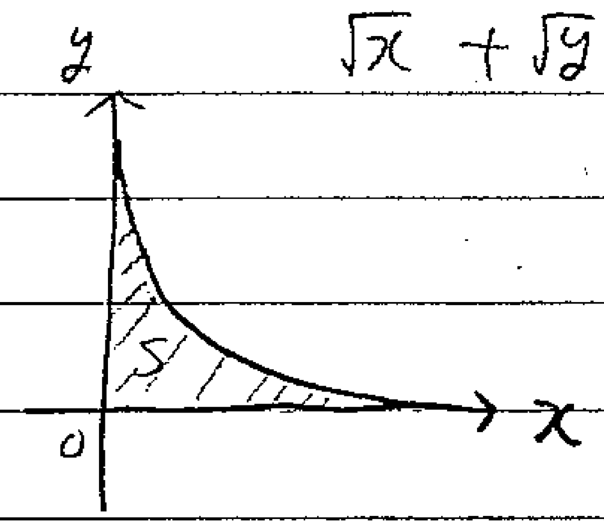
6/24 $\boxed{9}$

(1) $\sqrt{x}+\sqrt{y}=1$ の概形は下の図

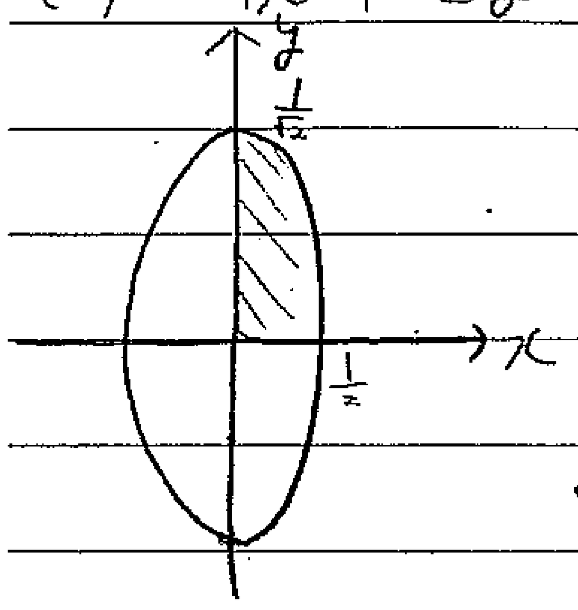


(与式) $\iff y=(1-\sqrt{x})^2$

以上より

$$S=\int_0^1(1-\sqrt{x})^2\,dx$$
$$=\int_0^1(1-2\sqrt{x}+x)\,dx$$
$$=\left[x-\frac{4}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2\right]_0^1$$
$$=1-\frac{4}{3}+\frac{1}{2}=\frac{1}{6}$$

(2) $4x^2+2y^2=1$



$x=\frac{1}{2}\cos\theta$

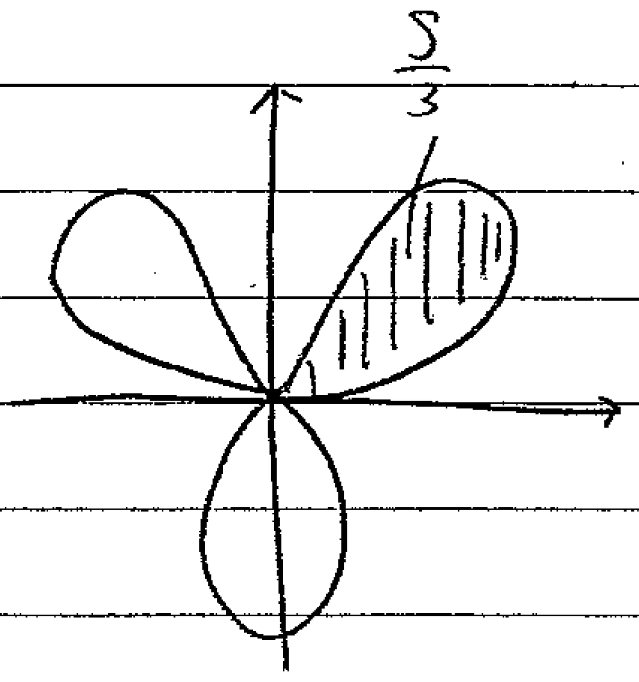
$y=\frac{1}{\sqrt{2}}\sin\theta$ とおくと

x	$0 \longrightarrow \frac{1}{2}$
θ	$\frac{\pi}{2} \longrightarrow 0$

より $0\leqq x\leqq\frac{1}{2}$ において

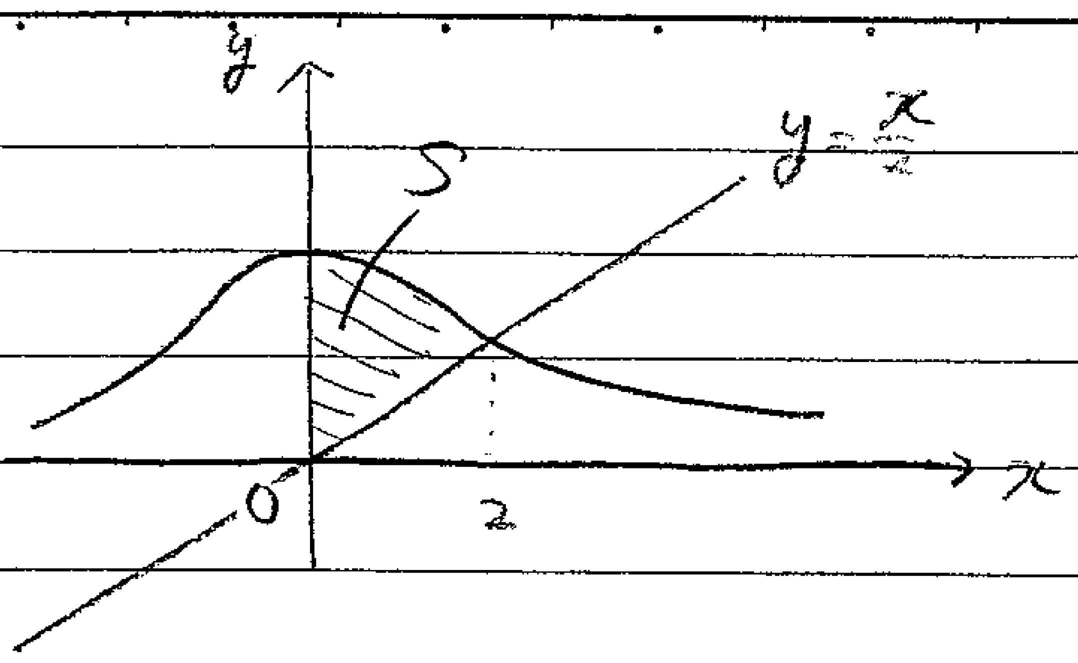
$$S=4\int_0^{\frac{1}{2}}y\,dx=4\int_{\frac{\pi}{2}}^0\frac{1}{\sqrt{2}}\sin\theta\times\left(-\frac{1}{2}\sin\theta\right)d\theta$$
$$=-\sqrt{2}\int_{\frac{\pi}{2}}^0\sin^2\theta\,d\theta$$
$$=-\sqrt{2}\left[\frac{1}{2}\theta-\frac{\sin2\theta}{4}\right]_{\frac{\pi}{2}}^0$$
$$=\frac{\pi}{2\sqrt{2}}$$

(3)



$$\frac{S}{3}=\frac{1}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\theta\,d\theta$$
$$\therefore S=\frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\theta\,d\theta$$
$$=\frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\,\frac{1-\cos6\theta}{2}\,d\theta$$
$$=\frac{3}{2}\cdot\frac{a^2}{2}\int_0^{\frac{\pi}{3}}(1-\cos6\theta)\,d\theta$$
$$=\frac{3}{2}\cdot\frac{a^2}{2}\left[\theta-\frac{1}{6}\sin6\theta\right]_0^{\frac{\pi}{3}}$$
$$=\frac{3}{2}\cdot\frac{a^2}{2}\cdot\frac{\pi}{3}$$
$$=\frac{a^2}{4}\pi$$

(4) $y=\frac{8}{x^2+4}$ と $y=\frac{1}{2}x$ の交点を求める



$\frac{x}{2}=\frac{8}{x^2+4}$

$\therefore\ x^3+4x-16=0$

$(x-2)(x^2+2x+8)=0$

$\therefore\ x=2$

$S=\int_0^2\left(\frac{8}{x^2+4}-\frac{x}{2}\right)dx$

$=\int_0^2\frac{8}{x^2+4}dx-\int_0^2\frac{x}{2}dx$

ここで $\int_0^2\frac{8}{x^2+4}dx$ について $x=2\tan\theta$ とおくと

x	$0\longrightarrow 2$
θ	$0\longrightarrow\frac{\pi}{4}$

$dx=\frac{1}{\cos^2\theta}d\theta$

よって $\int_0^2\frac{8}{x^2+4}dx$

$=\int_0^{\frac{\pi}{4}}\frac{2}{\tan^2\theta+1}\cdot 2\cdot\frac{1}{\cos^2\theta}d\theta$

$=\int_0^{\frac{\pi}{4}}4\,d\theta=\pi$ 、

よって $S=\pi-\int_0^2\frac{x}{2}dx$

$=\pi-\frac{1}{2}\left[\frac{1}{2}x^2\right]_0^2=\pi-1$

(5) $x^3+y^3-3xy=0$

$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ として変形すると

$r=\frac{3\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$

$S=\frac{1}{2}\int_0^{\frac{\pi}{2}}r^2d\theta$

$=\frac{9}{2}\int_0^{\frac{\pi}{2}}\frac{\tan^2\theta}{(1+\tan^3\theta)^2}\cdot\frac{1}{\cos^2\theta}d\theta$

$\tan\theta=t$ とおくと

$S=\frac{9}{2}\int_0^{+\infty}\frac{t^2}{(1+t^3)^2}dt$

$1+t^3=u$ とおいて

$S=\frac{3}{2}\int_1^{+\infty}\frac{1}{u^2}du=\frac{3}{2}\left[-\frac{1}{u}\right]_1^{+\infty}=\frac{3}{2}$

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(6) $y = a\cosh\frac{x}{a}$ より $\frac{dy}{dx} = \sinh\frac{x}{a}$

求める曲線の長さを L として

$$L = \int_0^h \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= \int_0^h \sqrt{1+\left(\sinh\frac{x}{a}\right)^2}\,dx$$
$$= \int_0^h \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}\,dx$$
$$= \left[\frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)\right]_0^h$$
$$= \frac{a}{2}\left(e^{\frac{h}{a}} - e^{-\frac{h}{a}}\right)$$

(7) 求める曲線の長さを L とする

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= \int_0^{2\pi} \sqrt{e^{2a\theta} + a^2 e^{2a\theta}}\,d\theta$$
$$= \sqrt{1+a^2}\int_0^{2\pi} e^{a\theta}\,d\theta$$
$$= \frac{\sqrt{1+a^2}}{a}\left[e^{a\theta}\right]_0^{2\pi}$$
$$= \frac{\sqrt{1+a^2}\left(e^{2\pi a} - 1\right)}{a}$$

(8) $y = 2\sqrt{px}$ 　 $\frac{dy}{dx} = \frac{p}{\sqrt{px}}$

$$S = 2\pi \int_0^1 y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 4\pi\sqrt{p}\int_0^1 \sqrt{x+p}\,dx$$
$$= 4\pi\sqrt{p}\left[\frac{2}{3}(x+p)^{\frac{3}{2}}\right]_0^1$$
$$= 4\pi\sqrt{p}\,\frac{2}{3}\left\{(p+1)^{\frac{3}{2}} - p^{\frac{3}{2}}\right\} = \frac{8}{3}\pi\sqrt{p}\left\{(p+1)^{\frac{3}{2}} - p^{\frac{3}{2}}\right\}$$

(9) $y = \frac{1}{2}(e^{x} + e^{-x})$

$$S = 2\pi \int_0^3 \frac{1}{2}(e^x + e^{-x})\sqrt{1 + \left\{\frac{1}{2}(e^x - e^{-x})\right\}^2}\, dx$$

$$= 2\pi \int_0^3 \frac{1}{2}(e^x + e^{-x})\sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}\, dx$$

$$= 2\pi \int_0^3 \frac{1}{2}(e^x + e^{-x})\sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\, dx$$

$$= 2\pi \int_0^3 \frac{1}{2}(e^x + e^{-x})\,\frac{1}{2}(e^x + e^{-x})\, dx$$

$$= \frac{\pi}{2}\int_0^3 (e^x + e^{-x})^2\, dx$$

$$= \frac{\pi}{2}\int_0^3 (e^{2x} + 2 + e^{-2x})\, dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^3$$

$$= \frac{\pi}{4}(e^6 + 12 - e^{-6})$$

(10) $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ $\begin{cases} \frac{dx}{dt} = a(1 - \cos t) \\ \frac{dy}{dt} = a\sin t \end{cases}$

$$S = 2\pi \int_\alpha^\beta y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^{2\pi} a(1 - \cos t)\sqrt{1 + \left\{\frac{a\sin t}{a(1 - \cos t)}\right\}^2} \cdot a(1 - \cos t)\, dt$$

$$= 2\pi a^2 \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}\, dt$$

$$= 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1 - \cos t)^{\frac{3}{2}}\, dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^3 \frac{t}{2}\, dt$$

$$= 8\pi a^2 \int_0^{2\pi} \sin\frac{t}{2}\left(1 - \cos^2\frac{t}{2}\right) dt$$

$$= 8\pi a^2 \left[-2\cos\frac{t}{2} + \frac{2}{3}\cos^3\frac{t}{2}\right]_0^{2\pi}$$

$$= 8\pi a^2 \left\{\left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)\right\}$$

$$= \frac{64}{3}\pi a^2$$