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$\boxed{8}$

(1) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)^2 \sin x \, dx$

$= \left[-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x \right]_0^{\frac{\pi}{2}}$

$= 1 - \frac{2}{3} + \frac{1}{5}$

$= \frac{8}{15}$

(2) $\int_1^2 x \log x \, dx$

$= \int_1^2 (\frac{1}{2}x^2)' \log x \, dx$

$= \left[\frac{1}{2}x^2 \log x \right]_1^2 - \int (\frac{1}{2}x^2)\frac{1}{x} \, dx$

$= \frac{1}{2} \times 4 \times \log 2 - \frac{1}{2} \times 1 \times \log 1) - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^2$

$= 2\log 2 - \frac{1}{2}(2 - \frac{1}{2})$

$= 2\log 2 - \frac{3}{4}$

(3) $\int_1^2 x\sqrt{4-x^2} \, dx$

$x = 2\sin\theta$ とおくと

$dx = 2\cos\theta \, d\theta$

x	$1 \longrightarrow 2$
θ	$\frac{\pi}{6} \longrightarrow \frac{\pi}{2}$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sin\theta\sqrt{4-4\sin^2\theta} \cdot 2\cos\theta \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\sin\theta\cos^2\theta \, d\theta$

$= \left[-\frac{8}{3}\cos^3\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \sqrt{3}$

(4) $\int_1^2 \sqrt{x^2-1} \, dx = I$ とすると

$I = \left[x\sqrt{x^2-1} \right]_1^2 - \int_1^2 \frac{x^2}{\sqrt{x^2-1}} \, dx$

$= \left[x\sqrt{x^2-1} \right]_1^2 - \int_1^2 \frac{x^2-1+1}{\sqrt{x^2-1}} \, dx$

$= \left[x\sqrt{x^2-1} \right]_1^2 - \int_1^2 \sqrt{x^2-1} \, dx - \int_1^2 \frac{1}{\sqrt{x^2-1}} \, dx$

$= \left[x\sqrt{x^2-1} \right]_1^2 - I - \int_1^2 \frac{1}{\sqrt{x^2-1}} \, dx$

$\therefore \ 2I = \left[x\sqrt{x^2-1} \right]_1^2 - \log \left[|x + \sqrt{x^2-1}| \right]_1^2$

$I = \frac{1}{2} \left[x\sqrt{x^2-1} \right]_1^2 - \log \left[|x+\sqrt{x^2-1}| \right]_1^2$

$= \frac{1}{2} \{ 2\sqrt{3} - \log(2+\sqrt{3}) \}$

(5) $\int_{-1}^{0} x\sqrt{x+1}\,dx$

$t=\sqrt{x+1}$ とおく

$t^2 = x+1$

$2t\,dt = dx$

x	$-1 \longrightarrow 0$
t	$0 \longrightarrow 1$

$(与式) = \int_0^1 (t^2-1)t\,2t\,dt$

$= 2\int_0^1 (t^4 - t^2)\,dt$

$= 2\left[\frac{1}{5}t^5 - \frac{1}{3}t^3\right]_0^1$

$= 2\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{4}{15}$

(6) (1) $(与式) = \lim_{n\to\infty} \frac{1}{n}\sum\left(\frac{k}{n}\right)^4$

$= \int_0^1 x^4\,dx$

$= \frac{1}{5}$

(2) $(与式) = \lim_{n\to\infty} \frac{1}{n}\sum\sqrt{1-\left(\frac{k}{n}\right)^2}$

$= \int_0^1 \sqrt{1-x^2}\,dx$

$x = \sin\theta$ $\quad dx = \cos\theta\,d\theta$ より

$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2\theta + 1)\,d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{4}$

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(7) $0<x<1$ に対して

$\frac{1}{\sqrt{2}\sqrt{1-x^2}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$

また $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ で

$x=\sin\theta$ とおくと　　$\begin{array}{c|c} x & 0 \longrightarrow 1 \\ \theta & 0 \longrightarrow \frac{\pi}{2} \end{array}$

$dx = \cos\theta\, d\theta$

$\therefore \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} d\theta$

$= \frac{\pi}{2}$

$\therefore \int_0^1 \frac{1}{\sqrt{2}\sqrt{1-x^2}}dx = \frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1+x^2)}} < \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$

$\therefore \frac{\pi}{2\sqrt{2}} < \int_0^1 \frac{1}{\sqrt{1-x^2}}dx < \frac{\pi}{2}$

(8) $f(x) = \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 x}}$ とすると

$0<x<\frac{\pi}{2}$ のとき　$0<\sin^2 x<1$ より

$1 < f(x) < \sqrt{2}$

この不等式を区間 $[0, \frac{\pi}{2}]$ で積分すると

$\int_0^{\frac{\pi}{2}} dx < \int_0^{\frac{\pi}{2}} f(x)dx < \int_0^{\frac{\pi}{2}} \sqrt{2}\, dx$

$\therefore \frac{\pi}{2} < \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{1}{2}\sin^2 x}}dx < \frac{\pi}{\sqrt{2}}$

(9) $0<x<1$ において次の不等式が成り立つ.

$\frac{1}{\sqrt{1+x^2}} \leqq \frac{1}{\sqrt{1+x^n}} \leqq 1$

$\Longleftrightarrow \int_0^1 \frac{dx}{\sqrt{1+x^2}} \leqq \int_0^1 \frac{1}{\sqrt{1+x^n}}dx < 1$

ここで $\int_0^1 \frac{1}{\sqrt{1+x^2}}dx = \left[\log|x+\sqrt{1+x^2}|\right]_0^1$

$= \log(1+\sqrt{2})$

よって $\log(1+\sqrt{2}) \leqq \int_0^1 \frac{dx}{\sqrt{1+x^n}} < 1$

(10) $y = \frac{\sqrt[n]{n!}}{n}$ とおくと

$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \frac{1}{n}\log\frac{n!}{n^n}$

$= \lim_{n\to\infty} \frac{1}{n}\left(\log\frac{1}{n} + \log\frac{2}{n} + \log\frac{3}{n} + \cdots \log\frac{n}{n}\right)$

$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\left(\frac{k}{n}\right)$

$= \int_0^1 \log x\, dx$

$= \left[x\log x - x\right]_0^1$

$= -1$

$\therefore \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}$