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(1) $\dfrac{1}{(x-1)^2(x-2)^3} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x-2} + \dfrac{D}{(x-2)^2} + \dfrac{E}{(x-2)^3}$ として

$\Leftrightarrow \quad 1 = (A+C)x^4 + (-7A+B-6C+D)x^3 + (18A-6B+13C-4D+E)x^2$

$+(-20A+12B-12C+5D-2E)x + (8A-8B+4C-2D+E)$ より

$A=-3 \quad B=-1 \quad C=3 \quad D=-2 \quad E=1$ より与式を積分して

$\log\left|\dfrac{x-2}{x-1}\right|^3 + \dfrac{1}{x-1} + \dfrac{2}{x-2} - \dfrac{1}{2(x-2)^2} + C$　C: 積分定数

(2) $\dfrac{1}{x(1+x^4)^2} = \dfrac{x^3}{x^4(1+x^4)^2}$

$x^4 = t$ とおくと $4x^3dx = dt$

$\displaystyle\int \frac{x^3}{x^4(1+x^4)^2}dx = \frac{1}{4}\int \frac{1}{t(1+t)^2}dt$

ここで $\dfrac{1}{t(1+t)^2} = \dfrac{1}{t} - \dfrac{1}{1+t} - \dfrac{1}{(1+t)^2}$ より

$\displaystyle\frac{1}{4}\int \frac{1}{t(1+t)^2} = \frac{1}{4}\int\left(\frac{1}{t} - \frac{1}{1+t} - \frac{1}{(1+t)^2}\right)dt$

$= \dfrac{1}{4}\left\{\log t - \log(1+t) + \dfrac{1}{1+t}\right\} + C$

$= \dfrac{1}{4}\left(\dfrac{1}{1+x^4} + \log\dfrac{x^4}{1+x^4}\right) + C$

(3) $\dfrac{1}{x^3+1} = \dfrac{1}{3}\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)$

$= \dfrac{1}{3}\left\{\dfrac{1}{x+1} - \dfrac{1}{2}\left(\dfrac{2x-1}{x^2-x+1} - \dfrac{3}{x^2-x+1}\right)\right\}$

$\displaystyle\int \frac{1}{x^2-x+1}dx = \int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx = \frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left\{\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right\}$

また $x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4} > 0$ より

$\displaystyle\int \frac{1}{x^3+1}dx = \frac{1}{3}\log|x+1| - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left\{\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right\} + C$ (Cは定数)

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4 $\frac{1}{x^4+1} = \frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1}$ とおく

計算して

$a=\frac{1}{2\sqrt{2}}$　$b=\frac{1}{2}$　$c=-\frac{1}{2\sqrt{2}}$　$d=\frac{1}{2}$

よって $\frac{1}{x^4+1} = \frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$

ここで $\int\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx = \frac{1}{2}\int\left(\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{(x+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}\right)dx$

$= \frac{1}{2}\log(x^2+\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x+1)$

同様に $\int\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx = \frac{1}{2}\log(x^2-\sqrt{2}x+1) - \tan^{-1}(\sqrt{2}x-1)$

よって $\int\frac{1}{x^4+1}dx = \frac{1}{4\sqrt{2}}\log\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\{\tan^{-1}(\sqrt{2}x+1) + \tan^{-1}(\sqrt{2}x-1)\} + C$

5 $\int\sec x\,dx = \int\frac{1}{\cos x}dx$

$t=\tan\frac{x}{2}$ とおくと

$\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$ $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$

$\Rightarrow \cos x = \frac{1-t^2}{1+t^2}$　$\sin x = \frac{2t}{1+t^2}$

$\frac{dt}{dx} = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2} = \frac{1+\tan^2\frac{x}{2}}{2}$

$\therefore dx = \frac{2}{1+t^2}dt$

$\therefore \int\sec x = \int\frac{1+t^2}{1-t^2}\cdot\frac{2}{1+t^2}dt$

$= \int\frac{2}{1-t^2}dt$

$= \int\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt$

$= \log\left|\frac{1+t}{1-t}\right| + C$

$= \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$　（Cは積分定数）

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(6) $\int \frac{1}{\cos x + \sin x} dx$ $\quad t = \tan\frac{x}{2}$ とおくと

$= \int \frac{1+t^2}{1+2t-t^2} \cdot \frac{2}{1+t^2} dt$

$= \int \frac{2}{1+2t-t^2} dt$

$= \int \frac{2}{(\sqrt{2})^2 - (t-1)^2} dt$

$= \frac{1}{\sqrt{2}} \int \left\{ \frac{1}{\sqrt{2}-(t-1)} + \frac{1}{\sqrt{2}+(t-1)} \right\} dt$

$= \frac{1}{\sqrt{2}} \{ -\log|\sqrt{2}-(t-1)| + \log|\sqrt{2}+(t-1)| + C$

$= \frac{1}{\sqrt{2}} \log\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right| + C$

$= \frac{1}{\sqrt{2}} \log\left|\frac{\tan\frac{x}{2}+\sqrt{2}-1}{\tan\frac{x}{2}-\sqrt{2}-1}\right| + C$ （C：積分定数）

(7) $\int \frac{\sin x}{3+\tan^2 x} dx = \int \frac{\sin x \cos^2 x}{3\cos^2 x + \sin^2 x} dx$

$= \int \frac{\cos^2 x}{1+2\cos^2 x} \cdot \sin x \, dx$ —— ①

ここで $t = \cos x$ とおく $\quad dt = -\sin x \, dx$ となるので

① $\Leftrightarrow -\int \frac{t^2}{1+2t^2} dt = -\int \frac{\frac{1}{2}(2t^2+1)-\frac{1}{2}}{2t^2+1} dt$

$= \frac{1}{2} \int \left(\frac{1}{2t^2+1} - 1 \right) dt$

$= \frac{1}{2} \int \left(\frac{1}{2} \cdot \frac{1}{t^2+\frac{1}{2}} - 1 \right) dt$ —— ①

ここで $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$ より

① $\Leftrightarrow \frac{\sqrt{2}}{4} \tan^{-1}\sqrt{2}t - \frac{1}{2}t + C$

$= \frac{\sqrt{2}}{4} \tan^{-1}(\sqrt{2}\cos x) - \frac{1}{2}\cos x + C$

(8) $\int \frac{1}{x+\sqrt{x-1}} dx$

$t = \sqrt{x-1}$ とすると $\quad x = t^2+1 \quad dx = 2t \, dt$

(与式) $= \int \frac{2t}{t^2+t+1} dt$

$= \int \frac{2t+1}{t^2+t+1} dt - \int \frac{1}{t^2+t+1} dt$

$= \log|t^2+t+1| - \int \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}} dt$

$= \log|t^2+t+1| - \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$= \log|t^2+t+1| - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$

$= \log|x+\sqrt{x-1}| - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right) + C$

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(9) $\displaystyle\int \frac{1}{(1-x)\sqrt{1+x+x^2}}\,dx$ ・・・・・・ (＊)

$t-x=\sqrt{1+x+x^2}$ とすると

$x=\dfrac{t^2-1}{2t+1}$

$\therefore\ dx=\dfrac{2(t^2+t+1)}{(2t+1)^2}dt$

$(＊)=\displaystyle\int \frac{1}{\left(1-\frac{t^2-1}{2t+1}\right)\left(t-\frac{t^2-1}{2t+1}\right)}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}dt$

$=\displaystyle\int \frac{-2}{t^2-2t-2}dt$

$=\dfrac{1}{\sqrt3}\displaystyle\int\left(\frac{1}{t-1+\sqrt3}-\frac{1}{t-1-\sqrt3}\right)dt$

$=\dfrac{1}{\sqrt3}\log\left|\dfrac{t-1+\sqrt3}{t-1-\sqrt3}\right|+C$

$(＊)=\dfrac{1}{\sqrt3}\log\left|\dfrac{x+\sqrt{x^2+x+1}-1+\sqrt3}{x+\sqrt{x^2+x+1}-1-\sqrt3}\right|+C$

(10) $\dfrac{x}{\sqrt{(x-a)(x-b)}}$ $(a<b)$

$t=\sqrt{\dfrac{x-a}{b-x}}$ とおくと　$x=\dfrac{a+bt^2}{t^2+1}$

$\dfrac{dt}{dx}=\dfrac{b-x+x-a}{2(x-a)^{\frac12}(b-x)^{\frac32}}$

$\therefore\ dx=\dfrac{2(b-x)\sqrt{(b-x)(x-a)}}{b-a}dt$ ・・・・ ①

$\displaystyle\int\frac{x}{\sqrt{(x-a)(b-x)}}dx$ ①より

$=\displaystyle\int\frac{2x(b-x)}{b-a}dt$

$=\displaystyle\int\frac{\frac{2(a+bt^2)(b-a)}{(t^2+1)^2}}{b-a}dt$

$=2\displaystyle\int\frac{bt^2+a}{(t^2+1)^2}dt$

$=\underbrace{2b\displaystyle\int\frac{dt}{t^2+1}}_{Ⓐ}+\underbrace{2\displaystyle\int\frac{a-b}{(t^2+1)^2}dt}_{Ⓑ}$

$Ⓐ=2b\displaystyle\int\frac{dt}{t^2+1}=\tan^{-1}t$

$=2b\tan^{-1}\sqrt{\dfrac{x-a}{b-x}}$

$Ⓑ=\displaystyle\int\frac{a-b}{(t^2+1)^2}dt$ について

$t=\tan\theta$　$\dfrac{dt}{d\theta}=\dfrac{1}{\cos^2\theta}$

よって　$2\displaystyle\int\frac{a-b}{(t^2+1)^2}dt$

$=(a-b)\left(\tan^{-1}\sqrt{\dfrac{x-a}{b-x}}+\dfrac{\sqrt{(x-a)(b-x)}}{b-a}\right)$

よって(与式) $=(a+b)\tan^{-1}\sqrt{\dfrac{a-x}{b-x}}-\sqrt{(x-a)(b-x)}+C$ （Cは定数）