平成 23 年度　機械系の数学演習 I　 第 7 回
問題

※　うｐ主が手計算で解いたものです。どうせ間違いだらけだろうと思いますが、おかしなところがあれば、各自で修正してくださいな(´・ω・｀)
明らかな間違いを発見された方は、wiki のコメント欄に書き込んでくだされば、皆さんのためになると思いますので、是非そうしてください。

1,　つぎの関数を積分せよ　：

$$\frac{1}{(x-1)^2(x-2)^3}$$

部分分数展開をする。

$$\frac{1}{(x-1)^2(x-2)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}+\frac{D}{(x-2)^2}+\frac{E}{(x-2)^3}$$ とおく。

$$\frac{1}{(x-1)^2(x-2)^3}=\frac{A(x-1)(x-2)^3+B(x-2)^3+C(x-1)^2(x-2)^2+D(x-1)^2(x-2)+E(x-1)^2}{(x-1)^2(x-2)^3}$$

$$1=A(x-1)(x-2)^3+B(x-2)^3+C(x-1)^2(x-2)^2+D(x-1)^2(x-2)+E(x-1)^2$$

$$f(x)=A(x-1)(x-2)^3+B(x-2)^3+C(x-1)^2(x-2)^2+D(x-1)^2(x-2)+E(x-1)^2=1$$

とおく。微分すると

$$\begin{aligned}f'(x)= \quad & A\{(x-2)^3+3(x-1)(x-2)^2\}\\ &+B\{3(x-2)^2\}\\ &+C\{2(x-1)(x-2)^2+2(x-1)^2(x-2)\}\\ &+D\{2(x-1)(x-2)+(x-1)^2\}\\ &+E\{2(x-1)\} \qquad =0\end{aligned}$$

$$\begin{aligned}f'(x)= \quad & A(x-2)^2(4x-5)\\ &+3B(x-2)^2\end{aligned}$$

$$+2C(x-1)(x-2)(2x-3)$$
$$+D(x-1)(3x-5)$$
$$+2E(x-1) \qquad = 0$$

∴

$$A(x-2)^2(4x-5)+3B(x-2)^2+2C(x-1)(x-2)(2x-3)+D(x-1)(3x-5)+2E(x-1)=0$$

さらに微分すると、

$$f''(x) = \quad A\{2(x-2)(4x-5)+(x-2)^2\cdot 4\}$$
$$+3B\{2(x-2)\}$$
$$+2C\{(x-2)(2x-3)+(x-1)(2x-3)+(x-1)(x-2)\cdot 2\}$$
$$+D\{(3x-5)+(x-1)\cdot 3\}$$
$$+2E \qquad = 0$$
$$f''(x) = \quad 2A(x-2)\{(4x-5)+(2x-4)\}$$
$$+6B(x-2)$$
$$+2C\{(2x^2-7x+6)+(2x^2-5x+3)+(2x^2-6x+4)\}$$
$$+D\{(3x-5)+(3x-3)\}$$
$$+2E \qquad = 0$$
$$f''(x) = \quad 2A(x-2)(6x-9)$$
$$+6B(x-2)$$
$$+2C(6x^2-18x+13)$$
$$+D(6x-8)$$
$$+2E \qquad = 0$$
$$f''(x) = \quad 6A(x-2)(2x-3)$$
$$+6B(x-2)$$
$$+2C(6x^2-18x+13)$$
$$+2D(3x-4)$$
$$+2E \qquad = 0$$

∴

$$6A(x-2)(2x-3)+6B(x-2)+2C(6x^2-18x+13)+2D(3x-4)+2E=0$$

これらより、

$$f(1)=0+B(-1)^3+0+0+0=1 \qquad \Rightarrow \qquad B=-1$$

$$f(2)=0+0+0+0+E=1 \qquad \Rightarrow \qquad E=1$$

$$f'(1)=A(-1)^3+B\cdot 3(-1)^2+0+0+0=0 \qquad \Rightarrow \qquad A+3B=0$$

$$f'(2)=0+0+0+D+2E=0 \qquad \Rightarrow \qquad D+2E=0$$

$$f''(2)=0+0+2C(24-36+13)+4D+2E=0 \quad \Rightarrow \qquad C+2D+E=0$$

まとめると、

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

行を並べ替えると、

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

はきだし法（線形代数学 I 参照）でこれを解く。

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&3\\0&0&0&1&2\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\\D\\E\end{bmatrix}=\begin{bmatrix}1&3&0&0&0\\0&1&0&0&0\\0&0&1&-2&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\0\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}A\\B\\C\\D\\E\end{bmatrix}=\begin{bmatrix}1&3&0&0&0\\0&1&0&0&0\\0&0&1&-2&3\\0&0&0&1&-2\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\0\\0\\1\end{bmatrix}$$

$$\therefore\begin{bmatrix}A\\B\\C\\D\\E\end{bmatrix}=\begin{bmatrix}1&3&0&0&0\\0&1&0&0&0\\0&0&1&-2&3\\0&0&0&1&-2\\0&0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}-3\\-1\\3\\-2\\1\end{bmatrix}$$

これより、

$$\frac{1}{(x-1)^2(x-2)^3}=\frac{-3}{x-1}+\frac{-1}{(x-1)^2}+\frac{3}{x-2}+\frac{-2}{(x-2)^2}+\frac{1}{(x-2)^3}$$

$$\int\frac{1}{(x-1)^2(x-2)^3}dx$$

$$=-3\int\frac{1}{x-1}dx-\int\frac{1}{(x-1)^2}dx+3\int\frac{1}{x-2}dx-2\int\frac{1}{(x-2)^2}dx+\int\frac{1}{(x-2)^3}dx$$

$$=-3\log|x-1|+\frac{1}{x-1}+3\log|x-2|+\frac{2}{x-2}-\frac{1}{2(x-2)^2}+C$$

2, つぎの関数を積分せよ :

$$\frac{1}{x(1+x^4)^2}$$

$$\int\frac{1}{x(1+x^4)^2}dx=\int\frac{4x^3}{4x^4(1+x^4)^2}dx$$

$1+x^4=t$ とおく。 $\frac{dt}{dx}=4x^3 \Rightarrow dt=4x^3dx$

$$\int\frac{1}{x(1+x^4)^2}dx=\int\frac{1}{4(t-1)t^2}dt=\frac{1}{4}\int\frac{1}{t^2(t-1)}dt$$

部分分数展開をする。

$\frac{1}{t^2(t-1)}=\frac{A}{t}+\frac{B}{t^2}+\frac{C}{t-1}$ とおく。

$$\frac{1}{t^2(t-1)}=\frac{At(t-1)+B(t-1)+Ct^2}{t^2(t-1)}$$

$$1=At(t-1)+B(t-1)+Ct^2$$

$f(t)=At(t-1)+B(t-1)+Ct^2=1$ とおく。

微分すると、

$$f'(t)=A\{(t-1)+t\}+B+2Ct=0 \Rightarrow f'(t)=A(2t-1)+B+2Ct=0$$

さらに微分すると

$$f''(t)=2A+2C=0 \Rightarrow f''(t)=A+C=0$$

これらより、

$$\begin{cases} f(0)=0-B+0=1 \\ f'(0)=-A+B+0=0 \\ f''(0)=A+C=0 \end{cases} \Rightarrow \begin{cases} B=-1 \\ A-B=0 \\ A+C=0 \end{cases} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

これを解くと、

$$\therefore \begin{bmatrix} A \\ B \\ C \end{bmatrix}=\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$\frac{1}{t^2(t-1)}=-\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t-1}$ であるから、

$$\int\frac{1}{x(1+x^4)^2}dx$$

$$=\frac{1}{4}\int\frac{1}{t^2(t-1)}dt$$

$$=\frac{1}{4}\int\left(-\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t-1}\right)dt$$
$$=\frac{1}{4}\left(-\int\frac{1}{t}dt-\int\frac{1}{t^2}dt+\int\frac{1}{t-1}dt\right)$$
$$=\frac{1}{4}\left(-\log|t|+\frac{1}{t}+\log|t-1|\right)+C$$
$$=\frac{1}{4}\left(-\log|1+x^4|+\frac{1}{1+x^4}+\log|x^4|\right)+C$$
$$=\frac{1}{4}\left\{-\log(1+x^4)+\frac{1}{1+x^4}+4\log|x|\right\}+C$$

3, つぎの関数を積分せよ :

$$\frac{1}{x^3+1}$$

$$\int\frac{1}{x^3+1}dx=\int\frac{1}{(x+1)(x^2-x+1)}dx$$

部分分数展開をする。

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$ とおく。

$$\frac{1}{(x+1)(x^2-x+1)}=\frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$1=A(x^2-x+1)+(Bx+C)(x+1)$$

$$f(x)=A(x^2-x+1)+(Bx+C)(x+1)=1$$ とおく。

両辺を微分すると、

$$f'(x)=A(2x-1)+B(x+1)+(Bx+C)=A(2x-1)+B(2x+1)+C=0$$

さらに両辺を微分すると、

$$f''(x)=2A+2B=0 \qquad \Rightarrow \qquad A+B=0$$

これらより、

$$\begin{cases} f(-1)=3A+0=1 \\ f'\left(\frac{1}{2}\right)=2B+C=0 \\ A+B=0 \end{cases} \Rightarrow \begin{cases} 3A=1 \\ 2B+C=0 \\ A+B=0 \end{cases} \Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

これを解くと、 $\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

したがって、 $\dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{1}{3}\left(\dfrac{1}{x+1} + \dfrac{-x+2}{x^2-x+1} \right)$ であるから、

$$\int \frac{1}{x^3+1}dx$$

$$= \frac{1}{3}\int \left(\frac{1}{x+1} + \frac{-x+2}{x^2-x+1} \right) dx$$

$$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{1}{2}\cdot\frac{2x-4}{x^2-x+1} \right) dx$$

$$= \frac{1}{3}\int \left\{ \frac{1}{x+1} - \frac{1}{2}\left(\frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1} \right) \right\} dx$$

$$= \frac{1}{3}\int \left(\frac{1}{x+1} - \frac{1}{2}\cdot\frac{2x-1}{x^2-x+1} + \frac{3}{2}\cdot\frac{1}{x^2-x+1} \right) dx$$

$$= \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2\left(x-\frac{1}{2}\right)}{\sqrt{3}} + C \qquad ・・・※$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

（※）公式　$\displaystyle\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C$　を用いた。

＜証明＞

$x = a\tan\theta \quad \left(-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}\right)$　とおく。

$$\frac{dx}{d\theta} = \frac{a}{\cos^2\theta} = a + a\tan^2\theta = a(1+x^2) \qquad \Rightarrow \qquad d\theta = \frac{1}{a(1+x^2)}dx$$

$$\int \frac{1}{x^2+a^2}dx$$

$$= \int \frac{1}{a^2\tan^2\theta + a^2}dx$$

$$= \frac{1}{a^2}\int \frac{1}{1+\tan^2\theta}dx$$

$$= \frac{1}{a}\int \frac{1}{a(1+x^2)}dx$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\arctan\frac{x}{a} + C \qquad \because \quad x = a\tan\theta \text{ より、} \tan\theta = \frac{x}{a} \quad \Rightarrow \quad \theta = \arctan\frac{x}{a}$$

4,　つぎの関数を積分せよ　：

$$\frac{1}{x^4+1}$$

$x^4+1 = \left(x^2+1\right)^2 - 2x^2 = \left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)$より、

$$\frac{1}{x^4+1} = \frac{1}{\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)} = \frac{1}{2\sqrt{2}}\left(\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right)$$

項ごとに不定積分すると、

$$\int \frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx = \frac{1}{2}\int \left\{ \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} \right\} dx$$

$$=\frac{1}{2}\left\{\log\left|x^2+\sqrt{2}x+1\right|-\sqrt{2}\sqrt{2}\arctan\left(\sqrt{2}x+1\right)\right\}$$

同様に

$$\int\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx=\frac{1}{2}\left\{\log\left|x^2-\sqrt{2}x+1\right|-\sqrt{2}\sqrt{2}\arctan\left(\sqrt{2}x-1\right)\right\}$$

であるから、

$$\int\frac{1}{x^4+1}dx=\frac{1}{4\sqrt{2}}\log\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+\frac{1}{2\sqrt{2}}\left\{\arctan\left(\sqrt{2}x+1\right)-\arctan\left(\sqrt{2}x-1\right)\right\}$$

5,　つぎの関数を積分せよ　：

$$\sec x \quad \left(=\frac{1}{\cos x}\right)$$

$$\int\sec x dx=\int\frac{1}{\cos x}dx=\int\frac{\cos x}{\cos^2 x}dx=\int\frac{\cos x}{1-\sin^2 x}dx$$

$t=\sin x$　とおく。

$$\frac{dt}{dx}=\cos x \qquad \Rightarrow \qquad dt=\cos x\ d$$

$$\int\sec x dx$$

$$=\int\frac{\cos x}{1-\sin^2 x}dx$$

$$=\int\frac{1}{1-t^2}dt$$

$$=-\int\frac{1}{(t+1)(t-1)}dt$$

$$=-\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\frac{1}{2}\int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$$

$$=\frac{1}{2}\left(\log|1+t|-\log|1-t|\right)+C$$

$$=\frac{1}{2}\left\{\log(1+\sin x)-\log(1-\sin x)\right\}+C$$

$$=\frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)+C$$

$$=\frac{1}{2}\log\left\{\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right\}+C$$

$$=\frac{1}{2}\log\left\{\frac{(1+\sin x)^2}{\cos^2 x}\right\}+C$$

$$=\frac{1}{2}\log\left|\frac{1+\sin x}{\cos x}\right|^2+C$$

$$=\log\left|\frac{1+\sin x}{\cos x}\right|+C$$

6,　つぎの関数を積分せよ　：

$$\frac{1}{\sin x+\cos x}$$

<置換積分(i)>

$$\frac{1}{\sin x+\cos x}=\frac{1}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}=\frac{1}{\sqrt{2}}\cdot\frac{\sin\left(x+\frac{\pi}{4}\right)}{\sin^2\left(x+\frac{\pi}{4}\right)}=-\frac{1}{\sqrt{2}}\cdot\frac{-\sin\left(x+\frac{\pi}{4}\right)}{1-\cos^2\left(x+\frac{\pi}{4}\right)}$$

$\cos\left(x+\frac{\pi}{4}\right)=t$ とおく。$\frac{dt}{dx}=-\sin\left(x+\frac{\pi}{4}\right)$ より、$dt=-\sin\left(x+\frac{\pi}{4}\right)dx$

$$\int\frac{1}{\sin x+\cos x}dx=\int-\frac{1}{\sqrt{2}}\cdot\frac{-\sin\left(x+\frac{\pi}{4}\right)}{1-\cos^2\left(x+\frac{\pi}{4}\right)}dx$$

$$=-\frac{1}{\sqrt{2}}\int\frac{1}{1-t^2}dt$$

$$=-\frac{1}{\sqrt{2}}\int\frac{1}{(1+t)(1-t)}dt$$

$$=-\frac{1}{2\sqrt{2}}\int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$$

$$=-\frac{1}{2\sqrt{2}}\int\left(\frac{1}{1+t}-\frac{-1}{1-t}\right)dt$$

$$=-\frac{1}{2\sqrt{2}}\{\log(1+t)-\log(1-t)\}+C$$

$$=-\frac{1}{2\sqrt{2}}\{\log(1+t)-\log(1-t)\}+C$$

$$=-\frac{1}{2\sqrt{2}}\log\left(\frac{1+t}{1-t}\right)+C$$

$$=-\frac{1}{2\sqrt{2}}\log\frac{1-t^2}{(1-t)^2}+C$$

$$=-\frac{1}{2\sqrt{2}}\log\frac{1-\cos^2\left(x+\frac{\pi}{4}\right)}{\left\{1-\cos\left(x+\frac{\pi}{4}\right)\right\}^2}+C$$

$$=-\frac{1}{2\sqrt{2}}\log\frac{\sin^2\left(x+\frac{\pi}{4}\right)}{\left\{1-\cos\left(x+\frac{\pi}{4}\right)\right\}^2}+C$$

$$=-\frac{1}{2\sqrt{2}}\log\left|\frac{\sin\left(x+\frac{\pi}{4}\right)}{1-\cos\left(x+\frac{\pi}{4}\right)}\right|^2+C$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{\sin\left(x+\frac{\pi}{4}\right)}{1-\cos\left(x+\frac{\pi}{4}\right)}\right|+C$$

$$=\frac{1}{\sqrt{2}}\log\left|\frac{1-\cos\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\right|+C$$

<置換積分(ii)>

$\tan\frac{x}{2}=t$ とおく。$\arctan t=\frac{x}{2}$ より、$dx=\frac{2}{1+t^2}dt$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\cdot\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\dfrac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)\cos^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

であるから、

$$\int\frac{1}{\sin x+\cos x}dx$$

$$=\int\frac{1}{\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$=\int\frac{2}{2t+1-t^2}dt$$

$$=-2\int\frac{1}{t^2-2t-1}dt$$

$$=-2\int\frac{1}{(t-1)^2-2}dt$$

$$=-2\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right| \qquad \because\ \int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| \quad (x\neq a)$$

$$=-\frac{1}{\sqrt{2}}\log\left|\frac{\tan\frac{x}{2}-1-\sqrt{2}}{\tan\frac{x}{2}-1+\sqrt{2}}\right|$$

$$=\frac{1}{\sqrt{2}}\log\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|$$

7,　つぎの関数を積分せよ　：

$$\frac{\sin x}{3+\tan^2 x}$$

$$\int \frac{\sin x}{3+\tan^2 x}dx = \int \frac{\sin x}{3+\dfrac{\sin^2 x}{\cos^2 x}}dx$$

$$= \int \frac{\sin x \cos^2 x}{3\cos^2 x + \sin^2 x}dx$$

$$= \int \frac{\cos^2 x}{2\cos^2 x + 1}\cdot \sin x dx$$

$t = -\cos x$ とおく。$dt = \sin x dx$

$$\int \frac{\sin x}{3+\tan^2 x}dx = \int \frac{(-t)^2}{2(-t)^2+1}dt$$

$$= \int \frac{t^2}{2t^2+1}dt$$

$$= \frac{1}{2}\int \frac{2t^2}{2t^2+1}dt$$

$$= \frac{1}{2}\left(\int dt - \int \frac{1}{2t^2+1}dt\right)$$

$$= \frac{1}{2}\left(t - \frac{1}{2}\int \frac{1}{t^2+\dfrac{1}{2}}dt\right)$$

$$= \frac{1}{2}\left\{t - \frac{1}{2}\int \frac{1}{t^2+\left(\dfrac{1}{\sqrt{2}}\right)^2}dt\right\}$$

$$= \frac{1}{2}\left\{t - \frac{1}{2}\cdot\sqrt{2}\arctan\left(\sqrt{2}t\right)\right\} \qquad \because \quad \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\frac{x}{a} \quad (x \neq a)$$

$$= \frac{1}{2}\left\{-\cos x - \frac{1}{\sqrt{2}}\arctan\left(-\sqrt{2}\cos x\right)\right\}$$

$$= \frac{1}{2}\left\{-\cos x + \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\cos x\right)\right\}$$

$$= -\frac{1}{2}\cos x + \frac{1}{2\sqrt{2}}\arctan\left(\sqrt{2}\cos x\right)$$

8, つぎの関数を積分せよ　:

$$\frac{1}{x+\sqrt{x-1}}$$

$t=\sqrt{x-1}$ とおく。$x=t^2+1$ であるから、$dx=2tdt$

$$\int\frac{1}{x+\sqrt{x-1}}dx=\int\frac{1}{(t^2+1)+t}\cdot 2tdt$$

$$=\int\frac{2t}{t^2+t+1}dt$$

$$=\int\frac{2t+1}{t^2+t+1}dt-\int\frac{1}{t^2+t+1}dt$$

$$=\int\frac{(t^2+t+1)'}{t^2+t+1}dt-\int\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

$$=\log(t^2+t+1)-\frac{2}{\sqrt{3}}\arctan\left\{\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right\}$$

9, つぎの関数を積分せよ　:

$$\frac{1}{(1-x)\sqrt{1+x+x^2}}$$

$t-x=\sqrt{1+x+x^2}$ とおく。$t^2-2tx=1+x$ であるから、$x=\dfrac{t^2-1}{2t+1}$ であり、

$$dx=\frac{(t^2-1)'(2t+1)-(t^2-1)(2t+1)'}{(2t+1)^2}dt=\frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}dt=\frac{2(t^2+t+1)}{(2t+1)^2}dt$$

$$\int\frac{1}{(1-x)\sqrt{1+x+x^2}}dx=\int\frac{1}{\left(1-\frac{t^2-1}{2t+1}\right)\left(t-\frac{t^2-1}{2t+1}\right)}\cdot\frac{2(t^2+t+1)}{(2t+1)^2}dt$$

$$=\int\frac{2(t^2+t+1)}{(2t+1-t^2+1)(2t^2+t-t^2+1)}dt$$

$$=\int\frac{2(t^2+t+1)}{(-t^2+2t+2)(t^2+t+1)}dt$$

$$=-\int\frac{2}{t^2-2t-2}dt$$

$$=-\int\frac{2}{(t-1)^2-3}dt$$

$$=-2\int\frac{1}{(t-1)^2-(\sqrt{3})^2}dt$$

$$=-2\cdot\frac{1}{\sqrt{3}}\log\left|\frac{t-1-\sqrt{3}}{t-1+\sqrt{3}}\right|$$

$$=-\frac{2}{\sqrt{3}}\log\left|\frac{x+\sqrt{1+x+x^2}-1-\sqrt{3}}{x+\sqrt{1+x+x^2}-1+\sqrt{3}}\right|$$

10,　つぎの関数を積分せよ　：

$$\frac{x}{\sqrt{(x-a)(b-x)}}\qquad(a<b)$$

$t=\sqrt{\dfrac{b-x}{x-a}}$ とおく。$x=\dfrac{at^2+b}{t^2+1}$ であるから、

$$dx=\frac{(at^2+b)'(t^2+1)-(at^2+b)(t^2+1)'}{(t^2+1)^2}dt$$

$$=\frac{2at(t^2+1)-(at^2+b)\cdot 2t}{(t^2+1)^2}$$

$$=\frac{(2at^3+2at)-(2at^3+2bt)}{(t^2+1)^2}$$

$$=\frac{2(a-b)t}{(t^2+1)^2}$$

$$\int \frac{x}{\sqrt{(x-a)(b-x)}}dx = \int \frac{\frac{at^2+b}{t^2+1}}{\left(\frac{at^2+b}{t^2+1}-a\right)\left(b-\frac{at^2+b}{t^2+1}\right)} \cdot \frac{2(a-b)t}{\left(t^2+1\right)^2}dt$$

$$= \int \frac{\left(at^2+b\right)\cdot 2(a-b)t}{\left\{at^2+b-a\left(t^2+1\right)\right\}\left\{b\left(t^2+1\right)-\left(at^2+b\right)\right\}}dt$$

$$= \int \frac{\left(at^2+b\right)\cdot 2(a-b)t}{\left(at^2+b-at^2-a\right)\left(bt^2+b-at^2-b\right)}dt$$

$$= \int \frac{\left(at^2+b\right)\cdot 2(a-b)t}{(b-a)\left(bt^2-at^2\right)}dt$$

$$= \int \frac{\left(at^2+b\right)\cdot 2(a-b)t}{(b-a)^2 t^2}dt$$

$$= \frac{2}{a-b}\int \frac{at^2+b}{t}dt$$

$$= \frac{2}{a-b}\int \left(at+\frac{b}{t}\right)dt$$

$$= \frac{2}{a-b}\left(\frac{1}{2}at^2+b\log|t|\right)+C$$

$$= \frac{2}{a-b}\left(\frac{1}{2}a\cdot\frac{b-x}{x-a}+b\log\left|\sqrt{\frac{b-x}{x-a}}\right|\right)+C$$

$$= \frac{2}{a-b}\left\{\frac{1}{2}a\cdot\frac{b-x}{x-a}+\frac{1}{2}b\log\left(\frac{b-x}{x-a}\right)\right\}+C$$

$$= \frac{1}{a-b}\left\{a\cdot\frac{b-x}{x-a}+b\cdot\log\left(\frac{b-x}{x-a}\right)\right\}+C$$