第6回

1. $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$+C$ (C:積分定数) $(a^2+b^2 \neq 0)$

(証明略)

より、

$$\int e^{-3x}\cos^2 x\,dx$$

$$= \int e^{-3x}\frac{1+\cos 2x}{2}dx$$

$$= -\frac{e^{-3x}}{6} + \frac{e^{-3x}}{26}(-3\cos 2x + 2\sin 2x) + C$$

$$= e^{-3x}\left(\frac{\sin 2x}{13} - \frac{3}{16}\cos 2x - \frac{1}{6}\right) + C$$

4. $\displaystyle\int x\tan^{-1}x\,dx$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2} + C$$

5. $\displaystyle\int\tan^3 x\,dx$

$$= \int\frac{\cos^2 x - 1}{\cos^3 x}\sin x\,dx$$

$$= \int\left(\frac{1}{\cos x} - \frac{1}{\cos^3 x}\right)\frac{d\cos x}{dx}dx$$

$$= \log|\cos x| + \frac{1}{2}\tan^2 x + C$$

8. $\displaystyle\int\frac{dx}{x+\sqrt{x^2-1}}$

$$= \int\left(x - \sqrt{x^2-1}\right)dx$$

$$= \frac{x^2}{2} - \frac{1}{2}\left(x\sqrt{x^2-1} - \log|x+\sqrt{x^2-1}|\right) + C$$

$$= \frac{1}{2}\left(x^2 - x\sqrt{x^2-1} + \log|x+\sqrt{x^2-1}|\right) + C$$

9. $\displaystyle\int x\sin^{-1}x\,dx$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\underbrace{\int\left(\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right)dx}_{I}$$

ここで、$x = \sin\theta$ とおいて、

$$I = \int\left(\frac{1}{\cos\theta} - \cos\theta\right)\cos\theta\,d\theta$$

$$= \int(1-\cos^2\theta)\,d\theta$$

$$= \int\frac{1-\cos 2\theta}{2}d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$$

ここで、$\theta = \sin^{-1}x$ より、

$\sin 2\theta = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$

$\therefore I = \dfrac{1}{2}\sin^{-1}x - \dfrac{x}{2}\sqrt{1-x^2} + C$

故に、(与式) $= \dfrac{x^2}{2}\sin^{-1}x - \dfrac{1}{4}\sin^{-1}x + \dfrac{x}{4}\sqrt{1-x^2} + C$

10. $\sin x = t$ $(-1<t<0,\ 0<t<1)$

とおいて、$\cos x\,dx = dt$

$$\int\frac{\cos^3 x}{\sin x}dx = \int\frac{(1-\sin^2 x)\cos x}{\sin x}dx$$

$$= \int\frac{1-t^2}{t}dt$$

$$= \int\left(\frac{1}{t} - t\right)dt$$

$$= \log|t| - \frac{1}{2}t^2 + C$$

$$= \log|\sin x| - \frac{1}{2}\sin^2 x + C$$

第7回

1. $\frac{1}{x^2(x-1)^2} = \frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$

$\int \frac{dx}{x^2(x-1)^2} = 2\log|x| - \frac{1}{x} - 2\log|x-1|$

$- \frac{1}{x-1} + C$ (C:積分定数)

$= 2\log\left|\frac{x}{x-1}\right| - \frac{2x-1}{x(x-1)} + C$

2. $\int \frac{x^3}{(x^2+1)^2}dx = \frac{1}{2}\int\frac{t-1}{t^2}dt$

$x^2+1=t$ とおいて, $= \frac{1}{2}\int\left(\frac{1}{t} - \frac{1}{t^2}\right)dt$

$2x\,dx = dt$ $= \frac{1}{2}\left(\log t + \frac{1}{t}\right)+C$

$= \frac{1}{2}\left\{\log(x^2+1)+\frac{1}{x^2+1}\right\}+C$

7. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int\frac{6t^5}{t^3+t^2}dt$

$t=\sqrt[6]{x}$ とおいて $= \int\frac{6t^3}{t+1}dt$

$dx=6t^5dt$ $= 6\int\left(t^2-t+1-\frac{1}{t+1}\right)dt$

$= 6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\log|t+1|\right)+C$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log|\sqrt[6]{x}+1|+C$

8. $\int\frac{dx}{(1-x)\sqrt{1+x}} = \int\frac{2}{2-t^2}dt$

$t=\sqrt{1+x}$ とおいて $= -\frac{1}{\sqrt{2}}\int\left(\frac{1}{t-\sqrt{2}}-\frac{1}{t+\sqrt{2}}\right)dt$

$t^2=1+x$

$2t\,dt=dx$ $= -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x}-\sqrt{2}}{\sqrt{1+x}+\sqrt{2}}\right|+C$

9. $\int\frac{dx}{\sqrt{(2-x)(3-x)}} = \int\frac{2}{2t-5}dt$

$t-x=\sqrt{(2-x)(3-x)}$ とおいて, $= 2\log|2t-5|\times\frac{1}{2}+C$

$dt = \frac{2\sqrt{(2-x)(3-x)}+2x-5}{2\sqrt{(2-x)(3-x)}}$ $= \log|2(x+\sqrt{(2-x)(3-x)})-5| + C$

第8回

2. $\int_0^1\frac{\log(1+x)}{1+x^2}dx = \int_0^{\frac{\pi}{4}}\frac{\log(1+\tan\theta)}{\cancel{1+\tan^2\theta}}\times\cancel{\sec^2\theta}\,d\theta$

$x=\tan\theta$ とおいて, $dx=\sec^2\theta\,d\theta$

$\begin{array}{c|c} x & 0\to 1 \\ \hline \theta & 0\to\frac{\pi}{4}\end{array}$

$= \int_0^{\frac{\pi}{4}}\log(1+\tan\theta)d\theta$

$= \int_0^{\frac{\pi}{4}}\log(\sin\theta+\cos\theta)d\theta - \int_0^{\frac{\pi}{4}}\log(\cos\theta)d\theta$

$= \int_0^{\frac{\pi}{4}}\log\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)d\theta - \int_0^{\frac{\pi}{4}}\log(\cos\theta)d\theta$

$\frac{\pi}{4}-\theta=t$ とおいて, $-d\theta=dt$

$\begin{array}{c|c} \theta & 0\to\frac{\pi}{4} \\ \hline t & \frac{\pi}{4}\to 0\end{array}$

$= \int_0^{\frac{\pi}{4}}\log\sqrt{2}\,d\theta + \int_0^{\frac{\pi}{4}}\log\cos\left(\frac{\pi}{4}-\theta\right)d\theta - \int_0^{\frac{\pi}{4}}\log\cos\theta\,d\theta$

$= \frac{\pi}{8}\log 2 + \cancel{\int_0^{\frac{\pi}{4}}\log\cos t\,dt} - \cancel{\int_0^{\frac{\pi}{4}}\log\cos\theta\,d\theta}$

$= \frac{\pi}{8}\log 2$

5. $\int_0^\pi\frac{x\sin x}{1+\cos^2x}dx = \int_0^\pi\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$

$= \int_0^\pi\frac{(\pi-x)\sin x}{1+\cos^2x}dx$

∴(与式) $= \frac{\pi}{2}\int_0^\pi\frac{\sin x}{1+\cos^2x}dx$

$\cos x=t$ とおく, $-\sin x\,dx=dt$ $= \frac{\pi}{2}\int_1^{-1}\frac{-1}{1+t^2}dt$

$\begin{array}{c|c} x & 0\to\pi \\ \hline t & 1\to -1\end{array}$ $= \pi\int_0^1\frac{dt}{1+t^2} = \frac{\pi^2}{4}$

6. $\int_1^2 x\log^2x\,dx$

$= \left[\frac{x^2}{2}\log^2x\right]_1^2 - \int_1^2\frac{x^2}{2}2(\log x)\frac{1}{x}dx$

$= 2\log^2 2 - \int_1^2 x\log x\,dx$

$= \left[\frac{1}{2}x^2\log x\right]_1^2 - \int_1^2\frac{1}{2}x^2\cdot\frac{1}{x}dx$

$= 2\log 2 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^2$

$= 2\log 2 - \frac{3}{4}$

∴(与式) $= 2\log^2 2 - 2\log 2 + \frac{3}{4}$

8.

(1) (与式) $= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\sqrt{1-\left(\frac{k}{n}\right)^2}}$

$= \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin^2 x}} d\theta = \frac{\pi}{2}$

(2) (与式) $= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}}$

$= \int_0^1 \sqrt{x}\, dx = \frac{2}{3}$

10. $y = \left\{\prod_{k=1}^{n} \left(1+\left(\frac{k}{n}\right)^2\right)\right\}^{\frac{1}{n}}$ とおいて、両辺対数をとる。

$\ln y = \ln \left\{\prod_{k=1}^{n} \left(1+\left(\frac{k}{n}\right)^2\right)\right\}^{\frac{1}{n}}$

$= \frac{1}{n} \sum_{k=1}^{n} \ln\left(1+\left(\frac{k}{n}\right)^2\right)$

$\lim_{n \to \infty} \ln y = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(1+\left(\frac{k}{n}\right)^2\right)$

$= \int_0^1 \log(1+x^2)\, dx$

$= \left[x \log(1+x^2)\right]_0^1 - \int_0^1 x \cdot \frac{2x}{1+x^2} dx$

$= \log 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$

$= \log 2 - \left[2x - 2\tan^{-1} x\right]_0^1$

$= \log 2 - 2 + \frac{\pi}{2}$

i.e. (与式) $= \exp\left(\ln 2 - 2 + \frac{\pi}{2}\right) = 2e^{\frac{\pi}{2}-2}$

第9回

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

より、$y^2 = a^2 - 3a^{\frac{4}{3}} x^{\frac{2}{3}} + 3a^{\frac{2}{3}} x^{\frac{4}{3}} - x^2$

$V = \pi \int_0^a y^2 dx$

$= \pi \left[a^2 x - \frac{9}{5} a^{\frac{4}{3}} x^{\frac{5}{3}} + \frac{9}{7} a^{\frac{2}{3}} x^{\frac{7}{3}} - \frac{x^3}{3}\right]_0^a$

$= \frac{16a^3}{105}\pi$

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a>0, b>0)$ より

$y^2 = b^2 - \frac{b^2}{a^2} x^2$, $x^2 = a^2 - \frac{a^2}{b^2} y^2$

$\frac{V_1}{2} = \pi \int_0^a y^2 dx = \pi \left[b^2 x - \frac{b^2}{3a^2} x^3\right]_0^a$

$= \frac{2}{3}\pi a b^2$

$\frac{V_2}{2} = \pi \int_0^b x^2 dy = \pi \left[a^2 y - \frac{a^2}{x^2} y^3\right]_0^b$

$= \frac{2}{3}\pi a^2 b$

$\therefore V_1 : V_2 = b : a$

8. $y = \log(\sin x)$ $\left(\frac{\pi}{3} \leq x \leq \frac{\pi}{2}\right)$

$l = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dt}{\sin t} = \left[\log\left|\tan\frac{t}{2}\right|\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -\log\frac{1}{\sqrt{3}}$

$= \frac{1}{2}\log 3$