6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$

$y^2 = a^2 - 3a^{\frac{4}{3}}x^{\frac{2}{3}} + 3a^{\frac{2}{3}}x^{\frac{4}{3}} - x^2$

体積 $V = \pi\int_0^a y^2 dx$

$= \pi\left[a^2x - \frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} + \frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}} - \frac{x^3}{3}\right]_0^a$

$= \frac{16a^3}{105}\pi$

7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a>0,\ b>0)$

$y^2 = b^2 - \frac{b^2}{a^2}x^2$ 、 $x^2 = a^2 - \frac{a^2}{b^2}y^2$

$\frac{V_1}{2} = \pi\int_0^a y^2 dx = \pi\left[b^2x - \frac{b^2}{3a^2}x^3\right]_0^a$

$= \frac{2}{3}\pi ab^2$

$\frac{V_2}{2} = \pi\int_0^b x^2 dy = \pi\left[a^2y - \frac{a^2}{3b^2}y^3\right]_0^b$

$= \frac{2}{3}\pi a^2 b$

$\therefore V_1 : V_2 = b : a$

8. $y = \log(\sin x) \quad \left(\frac{\pi}{3} \leq x \leq \frac{\pi}{2}\right)$

$l = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dt}{\sin t} = \left[\log\left|\tan\frac{t}{2}\right|\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -\log\frac{1}{\sqrt{3}} = \frac{1}{2}\log 3$

10. $S = 2\pi\int_0^{2\pi}\left[a(1-\cos t)\sqrt{1+\left(\frac{\sin t}{1-\cos t}\right)^2}\,a(1-\cos t)\right]dt$

$= 4\sqrt{2}\pi a^2\int_0^{\pi}(1-\cos t)^{\frac{3}{2}}dt$ (∵対称性)

$= 16\pi a^2\int_0^{\pi}\sin^3\frac{t}{2}dt$ (∵半角)

$= 16\pi a^2\int_0^{\pi}\sin\frac{t}{2}\left(1-\cos^2\frac{t}{2}\right)dt$

$= 16\pi a^2\left[-2\cos\frac{t}{2} + \frac{2}{3}\cos^3\frac{t}{2}\right]_0^{\pi}$

$= \frac{64}{3}\pi a^2$

第10回

1. $\lim_{h\to 1+0}\int_h^3 \frac{dx}{\sqrt{x^2-1}} = \left[\log\left|x+\sqrt{x^2-1}\right|\right]_1^3$

$= \log(3+2\sqrt{2})$

2. $\int_h^1 \frac{dx}{\sqrt{x+x^2}} = \int_h^1 \frac{dx}{\sqrt{(x+\frac{1}{2})^2-(\frac{1}{2})^2}}$

$= \left[\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|\right]_h^1$

$= \log\left(1+\frac{1}{2}+\sqrt{2}\right) - \log\left|h+\frac{1}{2}+\sqrt{h+h^2}\right|$

$\to \log(3+2\sqrt{2}) \quad (h\to+0)$

3. $\int_{-1+\varepsilon}^{1-\varepsilon'} \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}x\right]_{-1+\varepsilon}^{1-\varepsilon'}$

$= \sin^{-1}(1-\varepsilon') - \sin^{-1}(-1+\varepsilon)$

$\to \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi \quad (\varepsilon', \varepsilon \to +0)$

4.

$\int_0^1 x^5\log^2 x\,dx$

$= \frac{x^6}{6}\log^2 x - \int\frac{x^{\cancel{6}\,5}}{\cancel{6}\,3}\cdot 2(\log x)\frac{1}{\cancel{x}}dx$

$= \frac{x^6}{6}\log^2 x - \left(\frac{x^6}{18}\log x - \int\frac{x^5}{18}dx\right)$

$= \frac{x^6}{6}\log^2 x - \frac{x^6}{18}\log x + \frac{x^6}{108}$ ($x = e^{-u}$ とおいて $(u\to\infty)$ $(\because e^{-nu}(-u)\to 0)$

$\therefore$ (与式) $= \frac{1}{108}$ ($x^n\log x \to 0\ (x\to+0)$ から)

5. $\int_{-2}^{N} \frac{dx}{x^2+4} = \int_{-\frac{\pi}{4}}^{\tan^{-1}\frac{N}{2}} \frac{1}{4(\tan^2\theta+1)}\frac{2}{\cos^2\theta}d\theta$

$x = 2\tan\theta$ とおく

$dx = \frac{2}{\cos^2\theta}d\theta$

x	$-2 \to N$
θ	$-\frac{\pi}{4} \to \tan^{-1}\frac{N}{2}$

$= \frac{1}{2}\left[\theta\right]_{-\frac{\pi}{4}}^{\tan^{-1}\frac{N}{2}}$

$\to \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = \frac{3}{8}\pi \quad (N\to\infty)$

6. $\int \frac{\log x}{x^2}dx = -\frac{\log x}{x} + \int \frac{dx}{x^2}$

$= -\frac{\log x}{x} - \frac{1}{x}$

$[1, b]$ で積分して

$-\frac{\log b}{b} - \frac{1}{b} + 1 \to 1 \quad (b \to \infty)$

7. $\int \frac{dx}{e^x+2e^{-x}+3} = \int\left(\frac{1}{e^x+2} + \frac{1}{e^{-x}+1}\right)dx$

$e^x = t$ とおくと.
$e^x dx = dt$

$= \int\left(\frac{1}{t^2+2t} + \frac{1}{t+1}\right)dt$

$= \int\left(\frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+2}\right) + \frac{1}{t+1}\right)dt$

$\int^{t+2}\left(\frac{1}{t} - \frac{1}{t+2}\right)$

$= \left[\frac{1}{2}\log\frac{t}{t+2} + \log(t+1)\right]$

$\to \infty \quad (t \to \infty, -\infty)$

8. $\int_0^1 \sqrt{\frac{x}{1-x}}dx = \int_0^1 \frac{\sqrt{x}}{\sqrt{1-\sqrt{x}^2}}dx$

$\sqrt{x} = \sin t$ とおくと
$dx = 2\sin t\cos t\,dt$

$= \int_0^{\sin^{-1}\sqrt{1-\varepsilon}} \frac{2\sin^2 t\cos t}{\sqrt{1-\sin^2 t}}dt$

x	$0 \to 1-\varepsilon$
t	$0 \to \sin^{-1}\sqrt{1-\varepsilon}$

$\to \int_0^{\frac{\pi}{2}} 2\sin^2 t\,dt \quad (\varepsilon \to 0)$

$= \int_0^{\frac{\pi}{2}}(1-\cos 2t)\,dt$

$= \left[t - \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

9. $(a<b)$ $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} =$

$= \int_a^b \frac{dx}{\sqrt{-x^2+(a+b)x-ab}}$

$= \int_a^b \frac{dx}{\sqrt{-\left(x-\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 - ab}}$

$= \int_a^b \frac{dx}{\sqrt{-\left(x-\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2}}$

$x - \frac{a+b}{2} = t$ とおく.
$dx = dt$

x	$a \to b$
t	$\frac{a+b}{2}+\varepsilon \to \frac{b-a}{2}-\varepsilon'$

$= \lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0}} \int_{\frac{a-b}{2}+\varepsilon'}^{\frac{b-a}{2}-\varepsilon'} \frac{dt}{\sqrt{\left(\frac{a-b}{2}\right)^2 - t^2}}$

$= \left[\sin^{-1}\frac{2}{-a-b}t\right]_{\frac{a-b}{2}+\varepsilon}^{\frac{b-a}{2}-\varepsilon'}$

$= \sin^{-1}(1) - \sin^{-1}(-1)$

$= \frac{\pi}{2} - \frac{\pi}{2} = \pi$

10. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{1}{a^2+b^2\tan^2 x}\cdot\frac{1}{\cos^2 x}dx$

$x = \tan^{-1} t$ とおく.
$t = \tan x$
$dt = \frac{dx}{\cos^2 x}$

x	$0 \to \frac{\pi}{2}$
t	$0 \to N$

$= \lim_{N\to\infty} \int_0^N \frac{dt}{a^2+b^2t^2}$

$= \lim_{N\to\infty} \frac{1}{b^2}\int_0^N \frac{dt}{\left(\frac{b}{a}\right)^2 + t^2}$

$t = \frac{b}{a}\tan\theta$ とおく.
$dt = \frac{b}{a\cos^2\theta}d\theta$

t	$0 \to \infty$
θ	$0 \to \frac{\pi}{2}$

$= \frac{1}{b^2}\int_0^{\frac{\pi}{2}} \frac{\frac{b}{a}\cdot\frac{1}{\cos^2\theta}}{\left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\tan\theta\right)^2}d\theta$

$= \frac{1}{ab}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{2ab}$

[2. 別解]

$2x+1 = t$ とおくと.

$\int_1^3 \frac{dt}{\sqrt{t^2-1}}$ から以下同じ.

[5. 別解]

$\frac{x}{2} = t$ とおく.

$\int_{-1}^{\infty} \frac{1}{2}\frac{dt}{(1+t^2)} = \frac{1}{2}\left[\tan^{-1} t\right]_{-1}^{\infty}$

[7. 別解]

(与式) $= \int_a^b \left(\frac{e^x}{e^x+1} - \frac{e^x}{e^x+2}\right)dx$

$= \left[\log|e^x+1| - \log|e^x+2|\right]_a^b$

$= \log\frac{e^a+1}{e^a+2} - \log\frac{e^b+1}{e^b+2}$

$\to \log 2 \quad (a \to \infty \quad b \to -\infty)$