平成21年度 力学演習I.　　　　　　　　Date　.　.

問1

(1) $\mathbf{A}\cdot\mathbf{B} = -1+4+9 = 12$

(2) $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}\mathbf{k}$

$= (6-6)\mathbf{i} + (-3-3)\mathbf{j} + (2+2)\mathbf{k}$

$= -6\mathbf{j} + 4\mathbf{k}$

$\therefore \mathbf{A}\times\mathbf{B} = (0, -6, 4)$

(3) $\mathbf{e}_A = \dfrac{\mathbf{A}}{|\mathbf{A}|} = \dfrac{1}{\sqrt{14}}\mathbf{A} = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$

$\mathbf{e}_B = \left(-\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$

(4) $\mathbf{A}\times\mathbf{e}_B$ は $\mathbf{A}\perp\mathbf{A}\times\mathbf{e}_B$、$\mathbf{B}\perp\mathbf{A}\times\mathbf{e}_B$

ここで、$\mathbf{A}$と$\mathbf{B}$のなす角をθとすると

$|\mathbf{A}\times\mathbf{e}_B| = |\mathbf{A}|\sin\theta$

$\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$ は $\mathbf{B}\perp\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$、$\mathbf{A}\times\mathbf{e}_B\perp\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$

なので、$\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$ は $\mathbf{A}$、$\mathbf{B}$を含む平面上で$\mathbf{B}$と垂直なベクトルとなる

$|\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)| = A\sin\theta\cdot\sin 90^\circ = A\sin\theta$

$\mathbf{A}$を$\mathbf{B}$に垂直なベクトルに分解した大きさは $A\sin\theta$ なので、

$\mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$ は求める$\mathbf{B}$に垂直なベクトルである。

また、$\mathbf{A}\cdot\mathbf{e}_B = A\cos\theta$ で求める$\mathbf{B}$に平行なベクトルの大きさは$A\cos\theta$なので

$\mathbf{A} = (\mathbf{A}\cdot\mathbf{e}_B)\mathbf{e}_B + \mathbf{e}_B\times(\mathbf{A}\times\mathbf{e}_B)$

問2.

(1) $a = a_0 + a_1 t$

$v = \int_1^{-1} a\,dt = \int_1^{-1} (a_0 + a_1 t)\,dt$

$= a_0 t + \frac{1}{2} a_1 t^2 + C$ （C:積分定数）

$t = 0$ のとき $v(0) = v_0$ $\therefore v_0 = C$

$\therefore v = a_0 t + \frac{1}{2} a_1 t^2 + v_0$

$x = \int (a_0 t + \frac{1}{2} a_1 t^2 + v_0)\,dt$

$= \frac{1}{2} a_0 t^2 + \frac{1}{6} a_1 t^3 + v_0 t + C'$ （C':定数）

$t = 0$ の $x(0) = x_0$ $\therefore x_0 = C'$

$x = \frac{1}{2} a t^2 + \frac{1}{6} a t^3 + v_0 t + x_0$

(2) $x = Ae^{-Bt} \cos\omega t$

$v = \dot{x} = -ABe^{-Bt} \cos\omega t - A\omega e^{-Bt} \sin\omega t$

$a = \dot{v} = AB^2 e^{-Bt} \cos\omega t + AB\omega \sin\omega t + AB\omega e^{-Bt} \sin\omega t - A\omega^2 e^{-Bt} \cos\omega t$

$= -AB^2 e^{-Bt} \cos\omega t + 2AB \sin\omega t - A\omega^2 e^{-Bt} \cos\omega t$

$= Ae^{-Bt}(-B^2 \cos\omega t + 2B\omega \sin\omega t - \omega^2 \cos\omega t)$

(3) $a = p\frac{g}{r+x}$

$\frac{d^2x}{dt^2} = p\frac{g}{r+x}$

$x(t)$に関する微分方程式

$\frac{d^2x}{dt^2} = f(x)$ $\left(f(x) = p\frac{g}{r+x}\right)$

$\frac{d^2x}{dt^2} \cdot \frac{dx}{dt} = f(x)\frac{dx}{dt}$

両辺tについて積分して

$\int \frac{d^2x}{dt^2} \cdot \frac{dx}{dt}\,dt = \int f(x)\,dx$

$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \int p\frac{g}{r+x}\,dx$

ここで、$\frac{dx}{dt} = v$ から $\frac{1}{2}v^2 = pg \ln|r+x| + C$

$x(0) = 0$ 、$v(0) = 0$ 、$a(0) = \frac{pg}{r} > 0$ 、$x > 0$ 、$v > 0$ 、$a > 0$ 、$pg \ln|r| + C = 0$

$\therefore C = -pg \ln|r|$

$$\frac{1}{2}v^2 = pg\ln|r+x| - pg\ln|r|$$

$$\ln|r+x| = \frac{v^2}{2pg} + \ln r$$

$$r+x = e^{\frac{v^2}{2pg}+\ln r}$$

$$x = re^{\frac{v^2}{2pg}} - r$$

$$a = \frac{pg}{r+re^{\frac{v^2}{2pg}}-r} = \frac{pg}{re^{\frac{v^2}{2pg}}}$$

問3.

(1) x、y、z方向の速度、加速度をv_x、v_y、v_z、a_x、a_y、a_zとおく。

x方向：$ma_x = -mr(v_x - V_x)$

y方向：$ma_y = -mr(v_y - V_y)$

z方向：$ma_z = -mg$

(2) z方向について　$a_z = -g$

$$v_z(t) = \int a_z t = -gt + C \quad (C:定数)$$

$v_z(0) = B = C$ より $v_z(t) = -gt + B$

$$x_z(t) = \int v_z(t)\,dt = -\frac{1}{2}gt^2 + Bt + C' \quad (C':定数)$$

$x_z(0) = 0 = C'$ より $x_z(t) = -\frac{1}{2}gt^2 + Bt$

x方向について

$$\frac{dv_x(t)}{dt} = -r\{v_x(t) - V_x\}$$

$$\frac{1}{v_x(t) - V_x}\cdot\frac{dv_x(t)}{dt} = -r$$

両辺tで積分して

$$\int \frac{dV_x(t)}{v_x(t) - V_x} = -\int r\,dt$$

$$\ln|v_x(t) - V_x| = -rt + C \quad (C:定数)$$

$$\therefore v_x(t) = V_x + e^{-rt+C}$$

$$= V_x + De^{-rt} \quad (D = \pm e^C)$$

$v_x(0) = A = V_x + D$　$\therefore D = A - V_x$

より $v_x(t) = V_x + (A - V_x)e^{-rt}$

$$x(t) = \int v_x(t)\,dt = V_x t - \frac{1}{r}(A - V_x)e^{-rt} + C \quad (C:定数)$$

$x(0) = -\frac{1}{r}(A - V_x) + C = 0$　$C = \frac{1}{r}(A - V_x)$

$$x(t) = V_x t - \frac{1}{r}(A - V_x)e^{-rt} + \frac{1}{r}(A - V_x)$$

$$= V_x t + \frac{1}{r}(A - V_x)(1 - e^{-rt})$$

同様に、$v_y(t) = \overline{v}_y + D' e^{-rt}$（$D'$：定数）

$$v_y(0) = 0 = \overline{v}_y + D'$$

$$\therefore D' = -\overline{v}_y$$

$$v_y(t) = \overline{v}_y - \overline{v}_y e^{-rt}$$

$$y(t) = \int v_y(t)\,dt$$

$$= \overline{v}_y t + \frac{1}{r}\overline{v}_y e^{-rt} + C' \quad (C'\text{：定数})$$

$$y(0) = \frac{1}{r}\overline{v}_y + C' = 0$$

$$C' = -\frac{1}{r}\overline{v}_y$$

$$y(t) = \overline{v}_y t + \frac{1}{r}\overline{v}_y e^{-rt} - \frac{1}{r}\overline{v}_y$$

$$= \overline{v}_y t + \frac{1}{r}\overline{v}_y (e^{-rt} - 1)$$

(3) 着地までにかかる時間を T（$T>0$）とすると、

$z(T) = 0$ より、

$$-\frac{1}{2}gT^2 + BT = 0$$

$$-T\left(\frac{gT}{2} - B\right) = 0$$

$$\therefore T = \frac{2B}{g}$$

$x(T) = 0$ より、

$$0 = \overline{v}_x T - \frac{1}{r}Ae^{-rT} + \frac{1}{r}\overline{v}_x e^{-rt} + \frac{1}{r}A - \frac{1}{r}\overline{v}_x$$

$$(rT + e^{-rT} - 1)\overline{v}_x = Ae^{-rT} - A = -A(1 - e^{-rT})$$

$$T = \frac{2B}{g}$$

$$\left(\frac{2rB}{g} + e^{-\frac{2rB}{g}} - 1\right)\overline{v}_x = -A\left(1 - e^{-\frac{2rB}{g}}\right)$$

$$\left(2rB + ge^{-\frac{2rB}{g}} - g\right)\overline{v}_x = -Ag\left(1 - e^{-\frac{2rB}{g}}\right)$$

$$\overline{v}_x = -\frac{Ag\left(1 - e^{-\frac{2rB}{g}}\right)}{2rB + ge^{-\frac{2rB}{g}} - g}$$

問4

$$\begin{cases} x = \sin h \\ y = 1 - \cos h \\ z = h \end{cases}$$

(1) $s(h) = \int_0^h \sqrt{\left(\frac{dx}{dh}\right)^2 + \left(\frac{dy}{dh}\right)^2 + \left(\frac{dz}{dh}\right)^2}\, dh$

$= \int_0^h \sqrt{\cos^2 h + \sin^2 h + 1}\, dh$

$= \int_0^h \sqrt{2}\, dh = \sqrt{2}\, h$

(2) $\frac{dx}{dt} = \cos h \quad \frac{dy}{dt} = \sin h \quad \frac{dz}{dt} = 1$

$t(h) = \frac{1}{\sqrt{\cos^2 h + \sin^2 h + 1}} (\cos h、\sin h、1)$

$= \left(\frac{\cos h}{\sqrt{2}}、\frac{\sin h}{\sqrt{2}}、\frac{1}{\sqrt{2}}\right)$

(3) $ds = \sqrt{2}\, dh$

$\rho(h) = \frac{1}{\left|\frac{dt}{\sqrt{2}dh}\right|} = \frac{\sqrt{2}}{\left|\frac{dt}{dh}\right|}$

$\frac{dt}{dh} = \left(-\frac{\sin h}{\sqrt{2}}、\frac{\cos h}{\sqrt{2}}、0\right)$

$\left|\frac{dt}{dh}\right| = \left(\frac{-\sin h}{\sqrt{2}}\right)^2 + \left(\frac{\cos h}{\sqrt{2}}\right)^2$

$= \frac{1}{2}$

$\rho(h) = \frac{\sqrt{2}}{\frac{1}{2}} = 2\sqrt{2}$

$m\frac{d^2 z(t)}{dt^2} = -mg$

$\frac{d^2 z(t)}{dt^2} = -g$

$v_z(t) = \frac{dz(t)}{dt} = -gt + C$ (C:定数)

$z(0)=0 \quad \vec{n}(h)=\left(\frac{\cos h}{\sqrt{2}}、\frac{\sin h}{\sqrt{2}}、\frac{1}{\sqrt{2}}\right)$より

$v_z(0)=\frac{1}{\sqrt{2}}v_0$

$\therefore C=\frac{1}{\sqrt{2}}v_0$

$z(t)=\int v_z(t)\,dt=\int\left(-gt+\frac{1}{\sqrt{2}}v_0\right)dt$

$=-\frac{1}{2}gt^2+\frac{1}{\sqrt{2}}v_0t+C'$ (C':定数)

$z(0)=c=C'$

$z(t)=-\frac{1}{2}gt^2+\frac{1}{\sqrt{2}}v_0t$