

2.1 Strong form

Essentially all of the various numerical methods for solving the elastic equation of motion can be classified as particular implementations of the method of weighted residuals (MWR) (Finlayson 1972; Fletcher 1984). We represent the displacement, $\mathbf{u}(\mathbf{x})$, where u_i is the i component, and $\mathbf{x}$ is the spatial coordinate, as a linear combination of N linearly independent vector trial functions, $\{\phi_i^{(1)}(\mathbf{x}), \dots, \phi_i^{(N)}(\mathbf{x})\}$,

$$u_i(\mathbf{x}) = \sum_{n=1}^N c_n \phi_i^{(n)}(\mathbf{x}), \quad (1)$$

where the expansion coefficients c_n become the unknowns. Note that both $u_i(\mathbf{x})$ and c_n are functions of the angular frequency, ω , but we do not write this dependence explicitly. For the moment we assume that the basis is a complete set, which may be infinite. The conditions that must be satisfied by the trial functions, $\phi_i^{(n)}$, are discussed in the following. Unless the representation in eq. (1) fortuitously is an exact solution, the equation of motion and boundary conditions will not be satisfied exactly. The resulting errors are called the domain residuals and boundary residuals, respectively.

Typically, the trial functions for strong form MWR implementations are chosen so that they satisfy either the boundary and continuity conditions, or the partial-differential equation, but not both (e.g. Fletcher 1984). In the former case the boundary residual is zero; such MWR implementations are called domain methods. In the latter case the domain residual is zero; such MWR methods are called boundary methods. Finally, if the trial functions satisfy neither the differential equation nor the boundary and continuity conditions, both the domain and boundary residuals will be non-zero; the resulting MWR methods, which are called mixed methods, will not be considered further here. The work of Aki & Larner (1970) is an example of an MWR boundary method, and that of Spudich & Ascher (1983) is an example of an MWR domain method.

The word 'weighted' identifies the key concept in MWR. We choose a set of N linearly independent vector weight functions $w_i^{(m)}$ ($m=1, \dots, N$), which do not, in general, belong to the same function space as the trial functions, i.e. the continuity, differentiability and boundary conditions are in general different for the trial functions and the weight functions.

We multiply the domain residual (if we are considering a domain method) or the boundary residual (if we are considering a boundary method) by the weight functions and integrate to compute the weighted residuals. We then set the N weighted residuals to zero to obtain a set of N simultaneous linear equations for the expansion coefficients c_n .

For a domain method, the strong form of MWR for the elastic equation of motion in Cartesian coordinates in the frequency domain is

$$\int_V w_i^{(m)} [\rho \omega^2 u_i + (C_{ijkl} u_{k,l})_{,j} + f_i] dV = 0 \quad (m=1, \dots, N), \quad (2)$$

where ρ is the density, and $C_{ijkl} = C_{jikl} = C_{klij}$ are the 21 independent elastic constants. Note that the elastic moduli

C_{ijkl} are not restricted to being real or frequency independent; for example, the use of a superposition of standard linear solids to model anelastic attenuation (e.g. Liu, Anderson & Kanamori 1976) is entirely admissible. The subscript $,j$ denotes spatial differentiation with respect to the j coordinate, and f_i is the i component of the applied force. In eq. (2) the displacement u_i must be continuous, and the traction $(C_{ijkl} u_{k,l})_{,j}$ must be continuous in the n_j direction. As the displacement is an arbitrary superposition of the trial functions, each of the individual trial functions must satisfy these continuity conditions. Each individual trial function must also satisfy the boundary conditions, e.g. free or rigid boundary conditions. The weight functions, $w_i^{(m)}$, are not required to satisfy any differentiability, continuity, or boundary conditions; even δ functions are acceptable. We use the same number of weight functions and trial functions, so that the system of linear equations, eq. (2), will have a unique solution. For the moment we consider only media with homogeneous boundary conditions. Inhomogeneous boundary conditions (prescribed tractions or displacements) are discussed later.

2.2 Weak form

We now define a new operator, the weak form of the elastic equation of motion:

$$\begin{aligned} & \omega^2 \int_V y_i^{(m)} \rho v_i dV - \int_V y_{i,j}^{(m)} C_{ijkl} v_{k,l} dV \\ & = - \int_V y_i^{(m)} f_i dV. \quad (m=1, \dots, N). \end{aligned} \quad (3)$$

To emphasize that we have not yet established any connection between the weak-form operator, eq. (3), and the strong-form operator, eq. (2), we designate the displacement by v_i rather than u_i , and the weight functions by $y_i^{(m)}$ rather than $w_i^{(m)}$. The number of trial functions is equal to the number of weight functions.

The requirement for the existence of the integrals in eq. (3) is that both the solution v_i and the weight functions $y_i^{(m)}$ must be continuous, but no continuity requirements are placed on the derivatives of these functions. We expand v_i in terms of N linearly independent vector trial functions, $[\psi_i^{(1)}(\mathbf{x}), \dots, \psi_i^{(N)}(\mathbf{x})]$. At this point in our discussion we assume that the trial functions and weight functions are both complete sets, which may be infinite.

We represent the solution v_i as a linear combination of the trial functions

$$v_i(\mathbf{x}) = \sum_{n=1}^N d_n \psi_i^{(n)}(\mathbf{x}), \quad (4)$$

where the expansion coefficients d_n are the unknowns, and $\psi_i^{(n)}$ is the i component of the n th trial function.

Because, in general, the equivalence of the strong and weak solutions of partial-differential equations is a well-known result (e.g. Strang & Fix 1973, pp. 10–13; Johnson 1987, pp. 14–18; Dautray & Lions 1988, Ch. 7) the following discussion is a brief demonstration rather than a rigorous proof. To establish the relation between the weak-form operator defined in eq. (3) and the strong-form operator defined in eq. (2) we integrate eq. (3) by parts to

obtain

$$\int_V y_i^{(m)} [\rho \omega^2 v_i + (C_{ijkl} v_{k,l})_{,j} + f_i] dV - \int_S y_i^{(m)} [(C_{ijkl} v_{k,l}) n_j] dS = 0 \quad (m = 1, \dots, N), \quad (5)$$

where n_j is the outward unit normal vector.

As the weight functions $y_i^{(m)}$ are a complete set, their coefficients in eq. (5)—the terms in the square brackets—must be zero everywhere to guarantee that the right-hand side of eq. (5) is zero. The coefficient of $y_i^{(m)}$ in the volume integral must therefore be zero everywhere in the domain

$$[\rho \omega^2 v_i + (C_{ijkl} v_{k,l})_{,j} + f_i] = 0. \quad (6)$$

As this coefficient is in fact the strong form of the elastic equation of motion, we thus see that the strong form of the equation of motion is satisfied everywhere in the domain.

The preceding discussion assumed that the weak-form solutions were sufficiently differentiable to allow the integration by parts necessary to obtain eq. (5) from eq. (3). If, on the other hand, the trial functions do not satisfy the condition of continuity of traction, it might appear that the integration by parts cannot be performed. However, as was pointed out by Strang & Fix (1973, p. 11), trial functions whose tractions are not necessarily continuous (e.g. linear splines) can be obtained as the limit of a sequence of continuous functions whose tractions are continuous. Thus even when the integration by parts required to obtain eq. (5) cannot be rigorously performed, we can nevertheless regard this result as the limit of the result obtained for a sequence of admissible functions. This interpretation applies equally to other examples of integration by parts later in this section, but will not be repeated explicitly.

2.3 Natural and essential homogeneous boundary conditions

Using the same arguments applied in the preceding section to the volume integral in eq. (5), we can see that as the weight functions $y_i^{(m)}$ and the trial functions $\psi_i^{(n)}$ are complete sets, the integrand of the surface integral in eq. (5) must be zero everywhere on the boundary. This means that a free surface boundary condition

$$[(C_{ijkl} v_{k,l}) n_j] = 0 \quad (7)$$

is satisfied automatically everywhere on the boundary, without its having to be imposed explicitly on the trial functions (Strang & Fix 1973, p. 12).

The free boundary condition, eq. (7), is an example of a natural boundary condition, i.e. a boundary condition that will automatically be satisfied by the weak-form solution regardless of whether or not it is explicitly satisfied by each of the individual trial functions. In contrast, a fixed boundary condition is an essential boundary condition, i.e. a boundary condition that will not be satisfied by the weak-form solution unless it is explicitly satisfied by *all* of the trial functions and all of the weight functions on the part of the boundary where the displacement is required to be equal to zero (e.g. Courant 1943).

For the elastic equation of motion *any* homogeneous

boundary condition of the form

$$(C_{ijkl} v_{k,l}) n_j - S_{ij} v_j = 0, \quad (i = 1, 2, 3) \quad (8)$$

where S_{ij} are constants that can vary with position along the boundary and with frequency, is a natural boundary condition. We give examples in Appendix B of how to obtain the coefficients S_{ij} so that eq. (8) will correspond to a radiation boundary condition. We now add a surface integral to eq. (3) to define a weak-form operator for which the natural boundary condition is eq. (8), rather than eq. (7):

$$\omega^2 \int_V y_i^{(m)} \rho v_i dV - \int_V y_{i,j}^{(m)} C_{ijkl} v_{k,l} dV + \int_S y_i^{(m)} S_{ij} v_j dS = - \int_V y_i^{(m)} f_i dV \quad (m = 1, \dots, N). \quad (9)$$

To demonstrate that the natural boundary condition for eq. (9) is eq. (8), we integrate by parts, obtaining

$$\int_V y_i^{(m)} [\rho \omega^2 v_i + (C_{ijkl} v_{k,l})_{,j} + f_i] dV - \int_S y_i^{(m)} [(C_{ijkl} v_{k,l}) n_j - S_{ij} v_j] dS = 0 \quad (m = 1, \dots, N). \quad (10)$$

As $y_i^{(m)}$ is arbitrary, the natural boundary condition eq. (8) will automatically be satisfied by the weak-form solutions obtained from eq. (9) unless all of the trial functions satisfy a fixed boundary condition.

It is surprising that weak-form operators of the form of eq. (9) have not previously been used to solve the elastic equation of motion for media with general natural boundary conditions of the form eq. (8), as such augmented weak-form operators have been widely used in other fields. For example, Fix & Marin (1978) and Goldstine (1982) defined similar augmented weak-form operators for the Helmholtz equation (i.e. acoustic wave equation). Also, adding surface integrals to the weak-form operator to impose inhomogeneous boundary conditions (e.g. prescribed tractions) is extensively discussed in engineering (e.g. Strang & Fix 1973, pp. 70–71; Johnson 1987, pp. 40–41). Finally, note that Courant's classic (1943) paper presented an example in which the natural boundary condition was varied by adding a boundary integral to a weak-form operator.

2.4 Natural and essential continuity conditions

Continuity of displacement is an essential continuity condition, i.e. a continuity condition that must be explicitly satisfied by the weak-form solutions. This condition must therefore be satisfied by each of the trial functions. In contrast, continuity of traction is a natural continuity condition, i.e. a continuity condition which is automatically satisfied by the weak-form solutions without its having to be explicitly satisfied by each of the trial functions (Strang & Fix 1973, p. 14). We verify this using, for simplicity, the weak-form operator defined in eq. (3) rather than the weak-form operator defined in eq. (9); however, the following argument applies equally to either case.

As above, we integrate eq. (3) by parts. However, this time we divide the volume into two parts which are

separated by an internal surface S_1 . We obtain

$$\begin{aligned} & \int_V y_i^{(m)} [\rho \omega^2 v_i + (C_{ijkl} v_{k,l})_{,j} + f_i] dV \\ & - \int_{S_1} y_i^{(m)} [(C_{ijkl} v_{k,l}) n_j] dS \\ & - \int_{S_1} y_i^{(m)} [(C_{ijkl} v_{k,l})^+ - (C_{ijkl} v_{k,l})^-] n_j^+ dS = 0, \\ & (m = 1, \dots, N), \end{aligned} \quad (11)$$

where n_j is the outward unit normal vector to the outer surface and n_j^+ is the unit normal to the (arbitrarily chosen) 'upper' surface of S_1 , and $(C_{ijkl} v_{k,l})^+$ and $(C_{ijkl} v_{k,l})^-$ are evaluated on the 'upper' and 'lower' surfaces, respectively. Note that the sum of the separate two volume integrals is shown as $\int_V$ in eq. (11).

Considering only the integral on S_1 , we find that, as the weight functions $y_i^{(m)}$ are arbitrary, the following natural continuity condition (i.e. continuity of traction) will automatically be satisfied by the weak-form solutions at every point on S_1

$$[(C_{ijkl} v_{k,l})^+ - (C_{ijkl} v_{k,l})^-] n_j^+ = 0. \quad (12)$$

However, as the location of S_1 is arbitrary, continuity of traction will automatically be satisfied by the weak-form solutions everywhere in the medium. The fact that it is not necessary for the trial functions explicitly to satisfy continuity of traction is an important advantage of the weak form, particularly for laterally heterogeneous media.

Some previous workers (e.g. Wiggins 1976; Buland & Gilbert 1984) used the weak form to compute the normal modes of laterally homogeneous media, but used cubic splines as trial and weight functions so that continuity of traction could be explicitly satisfied. Note, however, that the extra effort required to explicitly force the traction to be continuous was unnecessary. Also, this approach cannot be easily generalized to laterally heterogeneous media, for which the effort required to explicitly force the traction to be continuous would be much greater.

2.5 Inhomogeneous boundary conditions

We now consider cases in which the traction or displacement on all or part of the outer boundary is required to be equal to a prescribed value. We obtain the corresponding weak-form operator by adding appropriate boundary integrals to the weak-form operator eq. (9). We consider a medium with a homogeneous boundary condition of the form eq. (8) imposed on a portion of the outer boundary, S_0 , an inhomogeneous traction boundary condition of the form

$$(C_{ijkl} v_{k,l}) n_j - \tau_i = 0, \quad (i = 1, 2, 3) \quad (13)$$

imposed on a portion of the outer boundary, S_τ , and an inhomogeneous displacement boundary condition of the form

$$v_i - D_i = 0, \quad (i = 1, 2, 3) \quad (14)$$

imposed on the remaining portion of the outer surface S_D . The weak-form operator that satisfies the above boundary

conditions is

$$\begin{aligned} & \omega^2 \int_V y_i^{(m)} \rho v_i dV - \int_V y_{i,j}^{(m)} C_{ijkl} v_{k,l} dV + \int_{S_0} y_i^{(m)} S_{ij} v_j dS \\ & = - \int_V y_i^{(m)} f_i dV - \int_{S_\tau} y_i^{(m)} \tau_i dS - \int_{S_D} y_{i,j}^{(m)} C_{ijkl} D_k n_l dS \\ & (m = 1, \dots, N). \end{aligned} \quad (15)$$

Equation (15) is essentially the weak-form version of the representation theorem.

To confirm that the solution of eq. (15) satisfies the boundary conditions on S_0 and S_τ we integrate by parts as follows

$$\begin{aligned} & \int_V y_i^{(m)} [\rho \omega^2 v_i + (C_{ijkl} v_{k,l})_{,j} + f_i] dV \\ & - \int_{S_0} y_i^{(m)} [(C_{ijkl} v_{k,l}) n_j - S_{ij} v_j] dS \\ & - \int_{S_\tau} y_i^{(m)} [(C_{ijkl} v_{k,l}) n_j - \tau_i] dS \\ & - \int_{S_D} y_i^{(m)} (C_{ijkl} v_{k,l}) n_j dS \\ & + \int_{S_D} y_{i,j}^{(m)} C_{ijkl} D_k n_l dS = 0 \quad (m = 1, \dots, N). \end{aligned} \quad (16)$$

As $y_i^{(m)}$ is arbitrary, the natural boundary condition eq. (8) will automatically be satisfied on S_0 by the weak-form solutions obtained from eq. (15). Similarly, we can see that the inhomogeneous boundary condition eq. (13) will be automatically satisfied on S_τ , and that the equation of motion will be satisfied everywhere in V . To confirm that the solution of eq. (15) satisfies eq. (14) on S_D , we integrate eq. (15) by parts as follows:

$$\begin{aligned} & \int_V v_i [\rho \omega^2 y_i^{(m)} + (C_{ijkl} y_{k,l}^{(m)})_{,j}] dV + \int_V y_i^{(m)} f_i dV \\ & - \int_{S_0} v_i C_{ijkl} y_{k,l}^{(m)} n_j dS + \int_{S_0} y_i^{(m)} S_{ij} v_j dS \\ & - \int_{S_\tau} v_i C_{ijkl} y_{k,l}^{(m)} n_j dS + \int_{S_\tau} y_i^{(m)} \tau_i dS \\ & - \int_{S_D} y_{i,j}^{(m)} C_{ijkl} n_l (v_k - D_k) dS = 0, \\ & (m = 1, \dots, N). \end{aligned} \quad (17)$$

As $y_{i,j}^{(m)} C_{ijkl} n_l$ is arbitrary, we see that $(v_k - D_k)$ must be zero everywhere on S_D . Thus we see that the inhomogeneous boundary conditions eqs (13) and (14) are natural boundary conditions of the augmented operator in eq. (15).

We now consider a dislocation. Suppose S_D^+ is the upper surface of the fault, S_D^- is the lower surface of the fault, and $\Delta_i = v_i^+ - v_i^-$ is the prescribed dislocation. We assume, for simplicity, that there is no surface with a prescribed traction, so that the surface integral over S_τ can be dropped

from eq. (15), which then becomes

$$\begin{aligned} \omega^2 \int_V y_i^{(m)} \rho v_i dV - \int_V y_{i,j}^{(m)} C_{ijkl} v_{k,l} dV + \int_{S_0} y_i^{(m)} S_{ij} v_j dS \\ = - \int_V y_i^{(m)} f_i dV + \int_{S_{D+}} y_{i,j}^{(m)} C_{ijkl} \Delta_k n_l dS \\ (m = 1, \dots, N). \end{aligned} \quad (18)$$

The force, f_i , for which $\int_V y_i^{(m)} f_i dV = \int_{S_{D+}} y_{i,j}^{(m)} C_{ijkl} \Delta_k n_l dS$ for all $y_i^{(m)}$ is defined to be the equivalent body force. f_i can be obtained from the right-hand side of eq. (18) essentially by inspection

$$f_i(\mathbf{x}) = - \int_{S_{D+}} [\delta(\mathbf{x} - \mathbf{x}_0)]_{,j} C_{ijkl}(\mathbf{x}_0) \Delta_k(\mathbf{x}_0) n_l(\mathbf{x}_0) dS(\mathbf{x}_0), \quad (19)$$

where we use $\mathbf{x}_0$ to denote the position on the surface S_{D+} .

The equivalent body force and the dislocation are equivalent in the following sense. We consider two separate solutions of eq. (18). In the first case, we drop the first term on the right-hand side of eq. (18) (i.e. the only inhomogeneous term is due to the dislocation), and in the second case we drop the second term on the right-hand side of eq. (18) and use the value of f_i given by eq. (19) in the first term of eq. (18) (i.e. the only inhomogeneous term is due to the equivalent body force). As the right-hand side of eq. (18) is equal for these two cases, the solutions of eq. (18) (and thus the synthetic seismograms everywhere in the medium) are equal for the two cases. Note that the classic paper by Burridge & Knopoff (1964) proved the equivalence of the body force defined by eq. (19) and the corresponding dislocation using the strong form of the elastic equation of motion. As is well known, eq. (19) can be used to show the equivalence of a shear dislocation and a superposition of double couples for an isotropic medium.

As inhomogeneous displacement or traction boundary conditions can always be replaced by the corresponding equivalent body forces, we will not include the last two terms on the right-hand side of eq. (15) in the results presented in the following sections. However, if desired, these additional inhomogeneous terms can be included in a straightforward manner.

2.6 Partial derivatives

To invert waveform data for earth structure it is necessary to be able to compute the partial derivatives of the synthetic seismograms with respect to changes in the model parameters. To do this we use the first-order Born approximation (e.g. Geller *et al.* 1990a; 1990b; Hara, Tsuboi & Geller 1991). Note that we are not restricted to treating the laterally heterogeneous part of the model as an 'infinitesimal' perturbation to a laterally homogeneous model, as the following results allow us to compute the partial derivatives for a laterally heterogeneous perturbation to a laterally heterogeneous initial model.

We make a perturbation δC_{ijkl} to the elastic moduli and a perturbation $\delta \rho$ to the density. We assume that the coefficients prescribing the natural boundary condition, S_{ij} , remain unchanged. We expand the perturbation to the

synthetic seismogram in terms of the trial functions

$$\delta v_i(\mathbf{x}) = \sum_{n=1}^N \delta d_n \psi_i^{(n)}(\mathbf{x}). \quad (20)$$

We obtain the weak form of the elastic equation of motion for the perturbed medium from eq. (9). [For simplicity we perturb eq. (9) rather than eq. (15), but the following derivation could equally well be applied to the latter.]

$$\begin{aligned} \omega^2 \int_V y_i^{(m)} (\rho + \delta \rho) (v_i + \delta v_i) dV \\ - \int_V y_{i,j}^{(m)} (C_{ijkl} + \delta C_{ijkl}) (v_{k,l} + \delta v_{k,l}) dV \\ + \int_S y_i^{(m)} S_{ij} (v_j + \delta v_j) dS \\ = - \int_V y_i^{(m)} f_i dV \quad (m = 1, \dots, N). \end{aligned} \quad (21)$$

Note that the unknowns in eq. (21) are the expansion coefficients δd_n in eq. (20). Using eq. (9) and retaining only first-order terms, eq. (21) becomes

$$\begin{aligned} \omega^2 \int_V y_i^{(m)} \rho \delta v_i dV - \int_V y_{i,j}^{(m)} C_{ijkl} \delta v_{k,l} dV \\ + \int_S y_i^{(m)} S_{ij} \delta v_j dS \\ = - \omega^2 \int_V y_i^{(m)} \delta \rho v_i dV + \int_V y_{i,j}^{(m)} \delta C_{ijkl} v_{k,l} dV \\ (m = 1, \dots, N). \end{aligned} \quad (22)$$

Eq. (22) is the weak form of the first-order Born approximation.

2.7 Finite approximation

The preceding discussion assumed that the trial and weight functions were complete sets, but all calculations will be limited to finite bases, which, in general, will not be complete sets. The accuracy and convergence of such finite approximations are discussed in detail by Strang & Fix (1973). Their most important result is Theorem 2.2 (p. 124), which states that in the limit as the basis becomes infinite, and thereby complete, if every element of the strong-form function space can be represented as a superposition of the weak-form trial functions, and if the system of linear equations (eq. 3, eq. 9, or eq. 15) is stably invertible, then the weak-form solution will converge to the strong-form solution. The factors controlling the rate of convergence are discussed by Strang & Fix (1973) and Johnson (1987).

3 FLUID MEDIA

Media containing both fluid and solid regions have posed problems for previous workers using weak-form methods. For example, Wiggins (1976) treated the fluid parts of the medium using the same formulation as for a solid, but setting the rigidity $\mu = 0$. This led to the existence of non-physical modes of vibration due to the extraneous degrees of freedom. Buland & Gilbert (1984, p. 111) used

the same basic approach as Wiggins, but set $\nabla \times \mathbf{u} = 0$ as an additional constraint on the displacement. This approach was apparently not completely successful in suppressing extraneous modes.

In contrast to most previous work, we use (a quantity proportional to) the pressure change as the dependent variable in the fluid regions, and displacement as the dependent variable in the solid regions. Arbitrary numbers of both fluid and solid regions are allowed, and their geometry can also be arbitrary. The continuity conditions at the fluid-solid boundaries are enforced by adding appropriate surface integrals to the weak-form operators for the respective regions. Note that Wu & Rochester (1990) used a somewhat similar approach in their study of core oscillations.

In the remainder of this paper we do not make an explicit distinction between the notation for the strong and weak forms. We use w to denote the weight functions, u to denote the displacement, ϕ to denote the trial functions, and c_n to denote the expansion coefficients of the trial functions. The reader is asked to keep in mind, however, that the weak- and strong-form weight and trial functions belong to different function spaces. We will continue to prove the equivalence of the weak and strong forms through the by now familiar procedure of integrating the weak form by parts.

We begin by considering the case of an isotropic fluid medium. For an isotropic fluid, in which the elastic moduli are equal to $C_{ijkl} = \lambda \delta_{ij} \delta_{kl}$, the strong form of the equation of motion, eq. (2) reduces to

$$\int_V w_i^{(m)} [\rho \omega^2 u_i + (\lambda u_{k,k})_{,i} + f_i] dV = 0 \quad (m = 1, \dots, N). \quad (23)$$

As the weight function $w_i^{(m)}$ is arbitrary, the term in square brackets in eq. (23) must be zero. We solve for u_i :

$$u_i = -\frac{(\lambda u_{k,k})_{,i} + f_i}{\rho \omega^2} \quad (24)$$

We substitute eq. (24) into the definition of the isotropic component of the stress tensor (i.e. the negative of the pressure change) to obtain

$$P = \lambda u_{k,k} = \lambda \left[\frac{(\lambda u_{k,k})_{,i} + f_i}{-\rho \omega^2} \right]_{,i} \quad (25)$$

Note that displacement appears in eq. (25) only in the form $\lambda u_{k,k}$. We now define a new scalar variable

$$Q = \frac{P}{\omega} = \frac{\lambda u_{k,k}}{\omega} \quad (26)$$

We combine eqs (24) and (26) to obtain an expression for the displacement when $f_i = 0$

$$u_i = -\frac{Q_{,i}}{\rho \omega} \quad (27)$$

We will use eq. (27) in the following when we consider the case of a fluid-solid boundary.

We substitute eq. (26) into eq. (25) to obtain, after a certain amount of manipulation,

$$\frac{\omega^2 Q}{\lambda} + \left[\frac{Q_{,i} + f_i/\omega}{\rho} \right]_{,i} = 0. \quad (28)$$

We have now obtained a scalar equation of motion. We express eq. (28) in MWR form, as

$$\int_V w^{(m)} \left\{ \frac{\omega^2 Q}{\lambda} + \left[\frac{(Q_{,i} + f_i/\omega)}{\rho} \right]_{,i} \right\} dV = 0 \quad (m = 1, \dots, N), \quad (29)$$

where $w^{(m)}$ is now a scalar weight function. Note that eq. (29) is equivalent to eq. (6) of Stephen (1988) when $f_i = 0$.

We represent the solution of eq. (29), Q , as a linear combination of N linearly independent scalar trial functions, $[\phi^{(1)}(\mathbf{x}), \dots, \phi^{(N)}(\mathbf{x})]$,

$$Q(\mathbf{x}) = \sum_{n=1}^N c_n \phi^{(n)}(\mathbf{x}), \quad (30)$$

where the expansion coefficients c_n are the unknowns in eq. (29). At this point in our discussion we assume that the trial functions and weight functions are both complete sets, which may be infinite.

The continuity conditions that must be satisfied by Q are continuity of Q itself, and, in a source-free region, continuity of $Q_{,i} n_i$ in the n_i direction. These conditions must be satisfied by each of the trial functions $\phi^{(n)}(\mathbf{x})$ used in eq. (30).

We now define a weak-form operator which is equivalent to eq. (29)

$$\int_V \left[\frac{\omega^2 w^{(m)} Q}{\lambda} - \frac{w^{(m)} (Q_{,i} + f_i/\omega)}{\rho} \right] dV = 0 \quad (m = 1, \dots, N). \quad (31)$$

In the same way as we obtained eq. (11) from eq. (3), we can, by integrating eq. (31) by parts, show that the continuity of $Q_{,i} n_i$ (which is equivalent to the continuity of the i component of displacement in the i direction) is a natural continuity condition which will be automatically satisfied by the weak-form solutions without its having to be explicitly satisfied by the trial functions. On the other hand, the continuity of Q (which is equivalent to the continuity of the pressure change) is an essential continuity condition, which must be satisfied by each of the trial functions in eq. (30). The essential and natural continuity conditions for a fluid medium are thus the opposite of those for a solid medium, for which continuity of displacement is an essential continuity condition and continuity of traction is a natural continuity condition.

To demonstrate the equivalence of the strong and weak forms we integrate eq. (31) by parts to obtain

$$\int_V w^{(m)} \left\{ \frac{\omega^2 Q}{\lambda} + \left[\frac{(Q_{,i} + f_i/\omega)}{\rho} \right]_{,i} \right\} dV - \int_S w^{(m)} \left[\frac{(Q_{,i} + f_i/\omega) n_i}{\rho} \right] dS = 0 \quad (m = 1, \dots, N). \quad (32)$$

The volume integral in eq. (32) is equivalent to the strong form, eq. (29). From the surface integral in eq. (32) we see that the natural boundary condition satisfied by eq. (31) at a force-free boundary is $Q_{,i} n_i = 0$. Eq. (24) shows that this is equivalent to $u_i n_i = 0$ in a source-free region. However, when we consider geophysical problems in media with a fluid region at the outer boundary we usually will want to have a traction-free surface, $Q = 0$. Note that this is an essential boundary condition for eq. (31).

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij}$$

$$P = \lambda u_{k,k}$$

3.1 Fluid-solid medium

We now define a weak-form operator for which the continuity conditions at a fluid-solid boundary are natural continuity conditions. For simplicity, the following discussion assumes that there is no applied body force at the fluid-solid boundary (i.e. $f_i = 0$). The continuity of normal displacement requires

$$\frac{Q_{,i} n_i^{(F)}}{\rho} - \omega u_i n_i^{(S)} = 0, \quad (33)$$

where Q and ρ are evaluated on the fluid side, u_i is the displacement on the solid side, and $n_i^{(F)}$ and $n_i^{(S)}$ are the outward unit normal vectors to the fluid and solid media, respectively. Note that $n_i^{(F)} = -n_i^{(S)}$. Also note that there are no restrictions on the geometry of the fluid-solid boundary.

The continuity of traction requires

$$\omega Q n_i^{(S)} - C_{ijkl} u_{k,l} n_j^{(S)} = 0, \quad (34)$$

where C_{ijkl} are the elastic moduli at the outer boundary of the solid medium, and other quantities are the same as in eq. (33). Note that eq. (34) implies that the tangential tractions at the fluid-solid boundary are zero.

We obtain a weak-form operator that satisfies the fluid-solid boundary conditions eqs (33) and (34) by adding appropriate surface integrals to the weak-form operators eqs (3) and (31)

$$\left. \begin{aligned} & \omega^2 \int_{V^{(S)}} w_i^{(m)} \rho u_i dV - \int_{V^{(S)}} w_{i,j}^{(m)} C_{ijkl} u_{k,l} dV \\ & + \int_{S^{(SF)}} \omega w_i^{(m)} Q n_i^{(S)} dS = - \int_{V^{(S)}} w_i^{(m)} f_i dV \\ & (m = 1, \dots, NS) \end{aligned} \right\} \text{Solid} \quad (35)$$

$$\left. \begin{aligned} & \int_{V^{(F)}} \left[\frac{\omega^2 w^{(m)} Q}{\lambda} - w_{,i}^{(m)} \frac{(Q_{,i} + f_i/\omega)}{\rho} \right] dV \\ & - \int_{S^{(FS)}} \omega w^{(m)} u_i n_i^{(F)} dS = 0 \\ & (m = NS + 1, \dots, NS + NF), \end{aligned} \right\} \text{Fluid}$$

where $V^{(S)}$ and $V^{(F)}$ are the solid and fluid volumes, respectively, $S^{(SF)}$ is the portion of the outer boundary of the solid medium at the solid-fluid interface, and $S^{(FS)}$ is the portion of the outer boundary of the fluid medium at the solid-fluid interface. NS and NF are the number of trial and weight functions in the solid and fluid media, respectively. The limits of summation in the respective trial function expansions must be appropriately modified. Note that the coupling between the solid and fluid media is accounted for entirely by the two surface integrals in eq. (35).

To demonstrate that eq. (35) satisfies the boundary conditions eqs (33) and (34), we integrate by parts to obtain

$$\begin{aligned} & \int_{V^{(S)}} w_i^{(m)} [\rho \omega^2 u_i + (C_{ijkl} u_{k,l})_{,j} + f_i] dV \\ & - \int_{S^{(SF)}} w_i^{(m)} [(C_{ijkl} u_{k,l}) n_j^{(S)} - \omega Q n_i^{(S)}] dS = 0 \\ & (m = 1, \dots, NS) \end{aligned} \quad (36)$$

$$\begin{aligned} & \int_{V^{(F)}} w^{(m)} \left\{ \frac{\omega^2 Q}{\lambda} + \left[\frac{(Q_{,i} + f_i/\omega)}{\rho} \right]_{,i} \right\} dV \\ & - \int_{S^{(FS)}} w^{(m)} \left[\frac{Q_{,i} n_i^{(F)}}{\rho} + \omega u_i n_i^{(F)} \right] dS = 0 \\ & (m = NS + 1, \dots, NS + NF). \end{aligned} \quad (37a)$$

These results can be generalized to a medium with an arbitrary number of fluid-solid boundaries in a straightforward manner. We do not give the explicit form of the weak-form operator for the partial derivatives, which is obtained following the same general approach as for the solid medium.

The above discussion assumed that the trial and weight functions were complete sets, but all calculations will be limited to finite bases. The discussion of this point in Section 2.7 also applies to the case of fluid-solid media.

4 DIRECT SOLUTION METHOD

We henceforth use the same finite set of functions as both trial functions and weight functions: $w_i^{(m)} = (\phi_i^{(m)})^*$ for a solid, and $w^{(m)} = (\phi^{(m)})^*$ for a fluid. Using the complex conjugates of the trial functions, rather than the trial functions themselves, as the weight functions enhances computational efficiency by minimizing the bandwidth of the linear equations (see Appendix A). The system of linear equations obtained using these weight and trial functions is called the Galerkin weak form of MWR.

We substitute the trial function expansion, eq. (4), into eq. (9) to obtain the Galerkin weak form of the elastic equation of motion for a solid medium with arbitrary natural boundary conditions

$$(\omega^2 \mathbf{T} - \mathbf{H} + \mathbf{R}) \mathbf{c} = -\mathbf{g}, \quad (37)$$

where $\mathbf{T}$ is the mass (kinetic energy) matrix, $\mathbf{H}$ is the stiffness (potential energy) matrix, $\mathbf{g}$ is the force vector, and $\mathbf{R}$ is the matrix operator corresponding to the natural boundary condition, eq. (8).

The DSM (Geller *et al.* 1990c) obtains the solution of the weak form of the equation of motion by directly solving the Galerkin weak form of the equation of motion, eq. (37), for a medium with a free surface natural boundary condition (i.e. $\mathbf{R} = 0$). In this paper we extend the DSM to media with arbitrary natural boundary conditions, for which, in general, $\mathbf{R} \neq 0$. We obtain the solution of the weak form of the equation of motion by directly solving eq. (37). In contrast, almost all previous work using the weak form computed synthetic seismograms by modal superposition. We discuss the relation of the DSM to the modal approach in the following.

The explicit form of the matrix elements and vector elements is as follows

$$T_{mn} = \int_V (\phi_i^{(m)})^* \rho \phi_i^{(n)} dV \quad (38)$$

$$H_{mn} = \int_V (\phi_{i,j}^{(m)})^* C_{ijkl} \phi_{k,l}^{(n)} dV \quad (39)$$

$$R_{mn} = \int_S (\phi_i^{(m)})^* S_{ij} \phi_j^{(n)} dS \quad (40)$$

$$g_m = \int_V (\phi_i^{(m)})^* f_i dV \quad (41)$$