4

[2] (5)

$$u(t,x)=\sum_{n=1}^{\infty} a_n \cos nt \sin nx$$

$(n\neq 1)$

$$a_n=\frac{2}{\pi}\int_0^\pi x^2(\pi-x)^2\sin x\sin nx\,dx$$

$$=\frac{-1}{\pi}\int_0^\pi x^2(\pi-x)^2(\cos(n+1)x-\cos(n-1)x)\,dx$$

$$=\frac{-1}{\pi}\int_0^\pi x^2(\pi-x)^2\cos(n+1)x\,dx+\frac{1}{\pi}\int_0^\pi x^2(\pi-x)^2\cos(n-1)x\,dx$$

$$=\frac{-12}{\pi(n-1)^4}\pi((-1)^{n+1}+1)$$

$$=\frac{-1}{\pi(n+1)}\int_0^\pi x^2(\pi-x)^2(\sin(n+1)x)'\,dx$$

$$=\frac{-1}{\pi(n+1)}\left(0-\int_0^\pi(2x(\pi-x)^2-2x^2(\pi-x))\sin(n+1)x\,dx\right)$$

$$=\frac{-2}{\pi(n+1)^2}\int_0^\pi x(\pi-x)(\pi-2x)(\cos(n+1)x)'\,dx$$

$$=\frac{-2}{\pi(n+1)^2}\left(0-\int_0^\pi((\pi-x)(\pi-2x)-x(\pi-2x)-2x(\pi-x))\cos(n+1)x\,dx\right)$$

$$=\frac{2}{\pi(n+1)^2}\int_0^\pi(\pi^2-3\pi x+2x^2-\pi x+2x^2-2\pi x+2x^2)\cos(n+1)x\,dx$$

$$=\frac{2}{\pi(n+1)^3}\int_0^\pi(\pi^2-6\pi x+6x^2)(\sin(n+1)x)'\,dx$$

$$=\frac{2}{\pi(n+1)^3}\left(0-\int_0^\pi(-6\pi+12x)\sin(n+1)x\,dx\right)$$

$$=\frac{+12}{\pi(n+1)^4}\int_0^\pi(-\pi+2x)(\cos(n+1)x)'\,dx$$

$$=\frac{12}{\pi(n+1)^4}\left(\pi(-1)^{n+1}+\pi-\int_0^\pi 2\cos(n+1)x\,dx\right)$$

$$=\frac{12}{\pi(n+1)^4}\left(\pi((-1)^{n+1}+1)-\frac{2}{n+1}[\sin(n+1)x]_0^\pi\right)$$

$$=\frac{12}{\pi(n+1)^4}\pi((-1)^{n+1}+1)$$

$$=\frac{12}{\pi(n-1)^4}\pi((-1)^n-1)-\frac{12}{\pi(n+1)^4}\pi((-1)^n-1)$$

$$=12((-1)^n-1)\left(\frac{1}{(n-1)^4}-\frac{1}{(n+1)^4}\right)$$

$$=12((-1)^n-1)\frac{n^4+4n^3+6n^2+4n+1-n^4+4n^3-6n^2+4n-1}{(n-1)^4(n+1)^4}$$

$$=96((-1)^n-1)\frac{n(n^2+1)}{(n^2-1)^4}=\frac{96n(n^2+1)}{(n^2-1)^4}((-1)^n-1)$$

$n=1$

$$a_1=\frac{-1}{\pi}\int_0^\pi x^2(\pi-x)^2\cos 2x\,dx+\frac{1}{\pi}\int_0^\pi x^2(\pi-x)^2\,dx$$

$$=\frac{12}{2^4}\cdot 2+\frac{1}{\pi}\int_0^\pi x^4-2x^3\pi+\pi^2x^2\,dx$$

$$=\frac{3}{2}+\frac{1}{\pi}\left[\frac{1}{5}x^5-\frac{1}{2}\pi x^4+\frac{1}{3}\pi^2x^3\right]_0^\pi$$

$$=\frac{3}{2}+\frac{1}{\pi}\left(\frac{1}{5}\pi^5-\frac{1}{2}\pi^5+\frac{1}{3}\pi^5\right)=\frac{3}{2}+\frac{1}{30}\pi^4$$

$\therefore u(t,x)$

$$=\left(\frac{1}{30}\pi^4+\frac{3}{2}\right)\cos t\sin x+\sum_{n=2}^{\infty}\frac{96n(n^2+1)}{(n^2-1)^4}((-1)^n-1)(\cos nt\sin nx)$$