②

$\sum_{n=1}^{\infty} a_n \sin nx \quad ((a_n \sin nx)\cos nt)'$

$= n a_n \sin(nx) \sin nt$

$0.$

(1) つづき

$\lambda > 0$ のとき

$\alpha^2 - \lambda = 0$

$\alpha = \pm\sqrt{\lambda}$

$T(t) = C_3 e^{\sqrt{\lambda} t} + C_4 e^{-\sqrt{\lambda} t}$

$T(t) = C_5 e^{\sqrt{\lambda} t} + C_6 e^{-\lambda t}$

$\alpha^2 + \lambda = 0$

$\alpha = \pm i\sqrt{\lambda}$

$X(x) = C_{10}\cos\sqrt{\lambda}x + C_8 \sin\sqrt{\lambda}x$

$= C_9 \cos\lambda x + C_{10}\sin\lambda x$

$= C_{11}(e^{i\lambda x} + e^{-i\lambda x}) + C_{12}(e^{i\lambda x} - e^{-i\lambda x})$

$= C_{13} e^{i\lambda x} + C_{14} e^{-i\lambda x}$

$u(t,x) = C_1 e^{\lambda(t+ix)} + C_2 e^{\lambda(t-ix)}$
$+ C_{15} e^{-\lambda(t+ix)} + C_{16} e^{-\lambda(t-ix)}$

$\lambda = 0$ のとき

$\alpha^2 = 0$

$\alpha = 0$

$T(t) = C_5 + C_6 t$

$X(x) = C_7 + C_8 x$

$u(t,x) = C_1 + C_2 t + C_3 x + C_4 t x$

$\lambda < 0$ のとき

$\alpha^2 - \lambda = 0$

$\alpha = \pm i\sqrt{-\lambda}$

$T(t) = C_5 \cos\sqrt{-\lambda}t + C_6 \sin\sqrt{-\lambda}t$

$= C_7 \cos(-\lambda t) + C_8 \sin(-\lambda t)$

$= C_9(e^{-i\lambda t} + e^{i\lambda t}) + C_{10}(e^{-i\lambda t} - e^{i\lambda t})$

$= C_{11} e^{-i\lambda t} + C_{12} e^{i\lambda t}$

$\alpha^2 + \lambda = 0$

$\alpha = \pm\sqrt{-\lambda}$

$X(x) = C_{13} e^{\sqrt{-\lambda}x} + C_{14} e^{\sqrt{-\lambda}x}$

$= C_{15} e^{-\lambda x} + C_{16} e^{\lambda x}$

$u(t,x) = C_7 e^{\lambda(-it-x)} + C_{18} e^{\lambda(-it+x)}$
$+ C_{19} e^{-\lambda(-it+x)} + C_{20} e^{-\lambda(-it-x)}$

$= C_1 e^{\lambda(t-ix)} + C_2 e^{\lambda(t+ix)}$
$+ C_3 e^{-\lambda(t+ix)} + C_4 e^{-\lambda(t-ix)}$

[2] (1) $u(t,x) = \sum_{n=1}^{\infty} a_n \cos nt \sin nx$

$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$

$= -\frac{1}{\pi}\int_0^{\pi} \cos(n+1)x - \cos(n-1)x \, dx$

$= -\frac{1}{\pi}\left[\frac{1}{n+1}\sin(n+1)x - \frac{1}{n-1}\sin(n-1)x\right]_0^{\pi}$

$= -\frac{1}{\pi}\left(\frac{1}{n+1}\sin(n+1)\pi - \frac{1}{n-1}\sin(n-1)\pi\right)$

$= -\frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\sin(n+1)\pi$

$= \frac{2}{\pi}\sin(n+1)\pi = 0 \ (n \neq 1)$

$a_1 = -\frac{1}{\pi}\int_0^{\pi} \cos 2x - 1 \, dx$

$= -\frac{1}{\pi}\left[\frac{1}{2}\sin 2x - x\right]_0^{\pi}$

$= 1$

$\therefore u(t,x) = \cos t \sin x$

(2) $a_n = -\frac{1}{\pi}\left(\frac{1}{n+2} - \frac{1}{n-2}\right)\sin(n+2)\pi$

$= \frac{4}{\pi}\sin(n+2)\pi = 0 \quad (n \neq 2)$

$a_2 = -\frac{1}{\pi}\int_0^{\pi} \cos 4x - 1 \, dx$

$= 1$

$\therefore u(t,x) = \cos 2t \sin 2x$

(3) $u(t,x) = \sum_{n=1}^{\infty} a_n \sin nt \sin nx$

key point

$\sum_{n=1}^{\infty} n a_n \sin nx = \sin 3x$ より、

$n a_n = \frac{2}{\pi}\int_0^{\pi} \sin 3x \sin nx \, dx$

$= \frac{6}{\pi}\sin(n+3)\pi = 0 \quad (n \neq 3)$

$3a_3 = -\frac{1}{\pi}\int_0^{\pi} \cos 6x - 1 \, dx = 1$

より、 $a_3 = \frac{1}{3}$

$\therefore u(t,x) = \frac{1}{3}\sin 3t \sin 3x$