COINCIDENCES AND FIXED POINTS OF SIX MAPS SATISFYING PROPERTY(E.A)

Anita Tomar, Government Degree College Dakpathar, Dehradun

anitatmr@yahoo.com

Abstract

The aim is to obtain coincidences and fixed points of six maps satisfying property (E.A) and integral type contractive condition using implicit relation due to their unifying power besides admitting new contraction condition.

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1. Introduction

Metrical common fixed point theorems generally require commutativity or its weaker form, some kind of continuity of one or more map, completeness and suitable containment of ranges of the involved maps besides an appropriate contraction condition to guarantee the existence of common fixed point. Researches in this domain are aimed at weakening one or more of these conditions. Weak commutativity of a pair of maps was introduced by Sessa [18] in fixed point considerations. There after number of generalizations of this notion have been obtained. Later on, Jungck [11] enlarged the class of noncommuting maps by compatible maps which asserts that a pair of selfmaps S and T of a metric space is compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. However the study of common fixed point of noncompatible maps is also equally interesting. The best example of noncompatible maps is found among pair of maps which are discontinuous at their common fixed point. Selfmaps S and T will be noncompatible if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either non zero or non existent. Also the concept of compatible maps was further improved by Jungck and Rhoades [13] with the notion of weakly compatible maps which merely commute at coincidence points. Selfmaps S and T of a metric space (X, d) are weakly compatible if $STx = TSx$ whenever $Sx = Tx$ for $x \in X$. However Singh [19] and Singh & Pant [21] used this concept without giving any name while establishing common fixed point theorem for maps on noncompatible spaces. For a brief development of weaker forms of commuting maps one may refer to Singh and Tomar [20].

In recent years, several common fixed point theorems for contractive type maps have been established by several authors (see, for instance, Jachymiski [10], Jungck et al. [12], Pant [14], Singh [19], Singh and Tomar [20], Tomar and Singh [22]). Using the concept of reciprocal continuity which is a weaker form of continuity of maps (see Pant [14], Popa ([16], [17]) some fixed point theorems satisfying certain implicit relations are proved. Hicks and Rhoades [7] established some common fixed point theorems in symmetric spaces using the fact that some of the properties of metric are not required in the proofs of certain metric theorems. In 2002 A. Branciari [5] introduced the notion of contractions of integral type and proved fixed point theorem for this class. Recently Aliouche et al. [3] established a common fixed point theorem for a pair of reciprocally continuous maps satisfying an implicit relation for integral type contractive condition. Pathak et al. [15] obtained a general common fixed point theorem of integral type for two pairs of weakly compatible maps satisfying integral type implicit relations in symmetric spaces by using the notion of a pair of maps satisfying property (E.A). Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that for all $x, y \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$.

A set X together with a symmetric d is called symmetric space.

If d is symmetric on a set X, then for $x \in X$ and $\varepsilon > 0$, we write $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. A topology $\tau(d)$ on X is given by $U \in \tau(d)$ if and only if for each $x \in X$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. A symmetric d is a semi-metric if for each $x \in X$ and for each $\varepsilon > 0$, $B(x, \varepsilon)$, is a neighbourhood of x in the topology $\tau(d)$.

A symmetric (respectively semi-metric) space (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric (respectively, semi-metric) d.

The difference of a symmetric and a metric comes from the triangle inequality. Actually a symmetric space need not be Hausdorff. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$.

The distinction between symmetric and a semi-metric is evident as one can easily construct a symmetric d such that $S(x, \varepsilon)$ need not be neighbourhood of x in $\tau(d)$. Recall, a subset S of a symmetric space (X, d) is said to be d-closed if for a sequence $\{x_n\}$ in S and a point $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $x \in S$. For a symmetric space (X, d), d-closedness implies $\tau(d)$ closedness and if d is a semi-metric, the converse is also true. A symmetric space (X, d) is said to be complete if every Cauchy Sequence in X converges to a point in X.

In order to obtain fixed point theorem of a symmetric space, we need some additional axioms. Wilson [23] gave the following axioms:

(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and a x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$.

Aliouche [2] gave the following axioms:

(HE) Given $\{x_n\}$, $\{y_n\}$, x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Pathak et al. [15] gave the following axioms:

(CE.1) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

(CE.2) Given $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ in X, $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\limsup_{n\to\infty} d(z_n, y_n) = \limsup_{n\to\infty} d(z_n, x_n)$.

Note that if d is metric, than (W.3), (W.4), (HE.), (CE.1), (CE.2) are automatically satisfied and if $\tau(d)$ is Hausdorff then (W.3) is satisfied. However Cho et al. [6] gave proposition to show that (W.4) $\Rightarrow$ (W.3) and (CE.1) $\Rightarrow$ (W.3) but reverse implication is not true. Also (CE.1.) is same as (CC) given by Cho et al. [6] .

Amri and Moutawakil [1] proved some common fixed point theorem under strict contractive conditions on a metric space for maps satisfying the property (E-A) which generalizes the concept of noncompatible maps in metric spaces. Recall that the pair (S, T) satisfies the property (E-A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Clearly both compatible and noncompatible pair enjoy (E-A) property. These definitions can be adapted to the setting of symmetric (semi-metric) spaces.

Let R_+ denote the set of nonnegative real numbers.

2. Implicit Relations.

In this paper we utilize implicit relations due to their versatility of deducing several contraction conditions at the same time. Let F_6 be the set of all continuous functions $F(t_1, t_2..., t_6) : R^6_+ \to R_+$ satisfying the following conditions:

$$(F_a)\ \int_0^{F(u, o, u, o, u, o)} \varphi(t)dt \leq 0 \text{ implies } u = 0;$$

$$(F_b)\ \int_0^{F(u, o, o, u, o, u)} \varphi(t)dt \leq 0 \text{ implies } u = 0.$$

The function $F(t_1, t_2, \ldots ,t_6) : R^6_+ \to R_+$ satisfies the conditions (F_1) if

$(F_1)\ \int_0^{F(u, u, o, o, u, u)} \varphi(t)dt > 0$ for all $u > 0$ and $\varphi : R_+ \to R$ is Lebesgue -integrable map which is summable.

Example 2.1 Let $F(t_1, t_2 ,\ldots ,t_6) := t_1 - c \min \{t_2, t_3, t_4, t, t_5, t_6\}$, where $0 < c < 1$ and $\varphi(t) = t^2$, for all t in R_+. Then

(F_a) $\int_0^{F(u, o, u, o, u, o)} t^2 dt \leq 0$; i.e., $\int_0^{u} t^2 dt \leq 0$,

which implies u = 0.

Similarly,

(F_b) $\int_0^{F(u, o, o, u, o, u)} t^2 dt \leq 0$; i.e., $\int_0^{u} t^2 dt \leq 0$,

which implies u = 0.

Further,

(F_1) $\int_0^{F(u, u, o, o, u, u)} t^2 dt = \int_0^{u} t^2 dt = \frac{u^3}{3} > 0$, for all $u > 0$.

So $F \in F_6$.

The main object of this paper is to extend and improve common fixed point theorem of Pathak et. al [15] satisfying certain integral type implicit relations, which are viable, productive and powerful tool in finding the existence of common fixed points of weakly compatible maps satisfying a contractive condition in symmetric spaces. (W.4), (CE.1) and (CE.2) are dropped. Our main result also demonstrates how several fixed point theorems can be unified using implicit relations.

Theorem 3.1. Let f, g, h, k, p and q be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W.3) and (HE). If one of p(X), q(X), hk(X) or fg(X) is a complete subspace of X such that

(i) $p(X) \subset fg(X)$, $q(X) \subset hk(X)$,

(ii) $\int_0^{F(d(px,\ qy),\ d(hkx,\ fgy),\ d(px,\ hkx),\ d(qy,\ fgy),\ d(px,\ fgy),\ d(hkx,\ qy))} \varphi(t) \leq 0$, for all x, y $\in$ X, where F $\in$ F_6 and satisfy properties (F_a), (F_b) and (F_1) and φ : $R_+ \to R$ is a Lebesgue - integrable map which is summable,

(iii) (p, hk) or (q, fg) satisfies property (E. A).

Then :

(I) p and hk have a coincidence.

(II) q and fg have a coincidence.

(III) p and hk have a common fixed point provided that they are weakly compatible.

(IV) q and fg have a common fixed point provided that they are weakly compatible.

(V) p, q, hk and fg have a common fixed point provided that (III) and (IV) both are true.

(VI) f, g, h, k, p and q have a unique common fixed point provided that k commutes with p and h and g commutes with f and q.

Proof. Let (q, fg) satisfies property (E.A). So that exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(qx_n\, t) = \lim_{n\to\infty} d(fgx_n, t) = 0$ for some t $\in$ X. By property (HE), we have $\lim_{n\to\infty} d(qx_n, fgx_n) = 0$. Since $q(X) \subset hk(X)$, then there exists a sequence $\{y_n\}$ in X such that $qx_n = hky_n$ for all $n \in N$.

Let $\lim_{n\to\infty} d(py_n, qx_n) = \varepsilon > 0$.

Using (ii) with $x = y_n$ and $y = x_n$, we have

$$\int_0^{F(d(py_n,\ qx_n),\ d(hky_n,\ fgx_n),\ d(py_n,\ hky_n),\ d(qx_n,\ fgx_n),\ d(py_n,\ fgx_n),\ d(hky_n,\ qy_n))} \varphi(t)\,dt \leq 0,$$

i.e, $$\int_0^{F(d(py_n,\ qx_n),\ d(qx_n,\ fgx_n),\ d(py_n,\ qx_n),\ d(qx_n,\ fgx_n),\ d(py_n,\ fgx_n),\ d(qx_n,\ qx_n))} \varphi(t)\,dt \leq 0.$$

Taking limit on as $n \to \infty$,

$$\int_0^{F(\varepsilon,\ 0,\ \varepsilon,\ 0,\ \varepsilon,\ 0)} \varphi(t)\,dt \leq 0,$$

which implies using the condition (F_a), $\varepsilon = 0$;

i.e., $\lim_{n\to\infty} d(py_n, qx_n) = 0$, i.e., $\lim_{n\to\infty} d(py_n, fgx_n) = 0$,

i.e., $\lim_{n\to\infty} py_n = \lim_{n\to\infty} qx_n = \lim_{n\to\infty} fgx_n = t$.

Suppose hk(X) is complete then $t = hku$ for some $u \in X$. Consequently, we have $\lim_{n\to\infty} py_n = \lim_{n\to\infty} qx_n = \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} hky_n = hku$.

If $pu \neq t$, we have using (ii)

$$\int_0^{F(d(pu,\ qx_n),\ d(hku,\ fgx_n),\ d(pu,\ hku),\ d(qx_n,\ fgx_n),\ d(pu,\ fgx_n),\ d(hku,\ qx_n))} \varphi(t)dt \leq 0.$$

Taking limit as $n \to \infty$,

$$\int_0^{F(d(pu,\ hku),\ 0,\ d(pu,\ hku),\ 0,\ d(pu,\ hku),\ 0)} \varphi(t)dt \leq 0,$$

which implies $d(pu, hku) = 0$, by using the condition (F_a).Thus $pu = hku = t$, i.e. p and hk have a coincidence.This proves (I).

Similarly by using $p(X) \subset fg(X)$, we get $pu = fgv = t$ for some $v \in X$.

If $qv \neq t$, using (ii) again, we have

$$\int_0^{F(d(pu,\ qv),\ d(hku,\ fgv),\ d(pu,\ hku),\ d(qv,\ fgv),\ d(pu,\ fgv),\ d(hku,\ qv))} \varphi(t)dt \leq 0,$$

i.e. , $\int_0^{F(d(t,\ qv),\ 0,\ 0,\ d(qv,\ t),\ 0,\ d(t,\ qv))} \varphi(t)dt \leq 0$,which implies $d(t, qv) = 0$ by using the condition (F_b),i.e., $qv = fgv = t$, i.e., q and fg have a coincidence. This prove (II).

Since the pair (p, hk) is weakly compatible, it follows that $p(hk)u = (hk)pu$,

i.e., $pu = hku$.

If $t \neq pu$, using (ii) we have

$$\int_0^{F(d(pt,\ qv),\ d(hkt,\ fgv),\ d(pt,\ hkt),\ d(qv,\ fgv),\ d(pt,\ fgv),\ d(hkt,\ qv))} \varphi(t)dt \leq 0;$$

i.e., $\int_0^{F(d(pt,\ t),\ d(pt,\ t),\ 0,\ 0,\ d(pt,\ t),\ d(pt,\ t))} \varphi(t)dt \leq 0$,

which contradict (F_1).Hence $d(pt, t) = 0$, i.e., $t = pt = hkt$, i.e. p and hk have a common fixed point.This proves (III).

Similarly, the weak compatibility of q and fg with (ii) yields $t = qt = fgt$; i.e., q and fg have a common fixed point. This proves (IV).

Thus t is a common fixed point of p, q, hk, and fg. This proves (V).

When fg(X) is assumed to be complete subspace of X, then the proof is similar. On the other hand the cases in which p(X) or q(X) is a complete subspace of X are, respectively, similar to the cases in which fg(X) or hk(X) is complete.

Next we shall show that kt = t. Taking x = ku and y = v in (ii)

$$\int_0^{F(d(pku,\ qv),\ d(hkku,\ fgv),\ d(pku,\ hkku),\ d(qv,\ fgv),\ d(pku,\ fgv),\ d(hkku,\ qv))} \varphi(t)\,dt \le 0.$$

Since k commutes with p and h and g commutes with f and q,

$$\int_0^{F(d(kpu,\ qv),\ d(khku,\ fgv),\ d(kpu,\ khku),\ d(qv,\ fgv),\ d(kpu,\ fgv),\ d(khku,\ qv))} \varphi(t)\,dt \le 0,$$

i.e. , $\int_0^{F(d(kt,\ t),\ d(kt,\ t),\ d(kt,\ kt),\ d(kt,\ t),\ d(kt,\ t))} \varphi(t)dt \le 0$,

i.e. , $\int_0^{F(d(kt,\ t),\ d(kt,\ t),\ 0,\ 0,\ d(kt,\ t),\ d(kt,\ t))} \varphi(t)dt \le 0$, which contradicts ($F_1$).

Hence d(kt, t) = 0, i.e., kt = t. Similarly we can prove that gt = t.

So, hkt = t and fgt = t implies ht = t and ft = t.

Hence ft = gt = ht = kt = pt = qt = t, i.e., q is a common fixed point of f, g, h, k, p and q.

For the uniqueness of common fixed point t, let w ≠ t be another common fixed point of f, g, h, k, p and q.

Then using (ii), we have

$$\int_0^{F(d(pt,\ qw),\ d(hkt,\ fgw),\ d(pt,\ hkt),\ d(qw,\ fgw),\ d(pt,\ fgw),\ d(hkt,\ qw))} \varphi(t)dt \le 0,$$

i.e. $\int_0^{F(d(t,\ w),\ d(t,\ w),\ d(t,\ t),\ d(w,\ w),\ d(t,\ w),\ d(t,\ w))} \varphi(t)dt \le 0$,

i.e., $\int_0^{F(d(t,\ w),\ d(t,\ w),\ 0,\ 0,\ d(t,\ w),\ d(t,\ w))} \varphi(t)dt \le 0$.

which contradict (F_1). Therefore, d (t, w) = 0, i.e. ,t = w

Hence t is the unique common fixed point of f, g, h, k, p and q.

Corollary 3.1. Let A, B, S and T be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W. 3) and (HE). If one of AX, BX, SX or TX is a complete subspace of X such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) $\int_0^{F(d(Ax,\ By),\ d(Sx,\ Ty),\ d(Ax,\ Sx),\ d(By,\ Ty),\ d(Ax,\ Ty),\ d(Sx,\ Ty))} \varphi(t)\,dt \leq 0$, for all $x, y \in X$, where $F \in F_6$ and satisfy properties (F_a), (F_b) and (F_1) and $\varphi : R_+ \to R$ is a Lebesgue – integrable map which is summable,

(iii) (A, S) or (B, T) satisfies property (E.A).

Then :

(I) A and S have a coincidence

(II) B and T have a coincidence.

(III) A and S have a common fixed point provided that they are weakly compatible.

(IV) B and T have a common fixed point provided they are weakly compatible.

(V) A, B, S and T have unique common fixed point provided that (III) and (IV) both are true.

Proof. Proof is similar to theorem 3.1 by substituting $g = k = I$, the identity map, $p = A$, $q = B$, $f = T$ and $h = S$.

Remark 3.1. Corollary 3.1 is the improved form of main result of Pathak et al. [15]

Corollary 3.2. Let A, B, S and T be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W. 3) and (H.E). If one of AX, BX, SX or TX is a complete subspace of X such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) $F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By) \leq 0$, for all $x, y \in X$, where $F \in F_6$ and satisfy properties (Fa), (F_b) and (F_1),

(iii) (A, S) or (B, T) satisfies property (E.A).

Then :

(I) A and S have a coincidence.

(II) B and T have a coincidence.

(III) A and S have a common fixed point provided that they are weakly compatible.

(IV) B and T have a common fixed point provided they are weakly compatible.

(V) A, B, S and T have unique common fixed point provided that (III) and (IV) both are true.

Proof. Substitute $\varphi(t) = 1$, $g = k = I$, the identity map ,$p = A$, $q = B$, $h = S$ and $f = T$ in theorem 3.1.

Remark 3.2. Corollary 3.2 is the improved form of results of Popa [17] and Imdad et al. [8] and corollary 3.1 of Pathak et al. [15].

Corollary 3.3. Let A, B, S and T be selfmaps of a metric space (X, d). If one of AX, BX, SX or TX is a complete substance of X such that

(i) $SX \subset BX$ and $TX \subset AX$,

(ii) $F(d(Sx, Ty), d(Ax, By), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ax, Ty)) \leq 0$,
for all $x, y \in X$, where $F \in F_6$ and satisfy property (F_a), (F_b) and (F_1).

(iii) (S, A) or (T, B) satisfies property (E.A).

Then :

I. A and S have a coincidence.

II. B and T have a coincidence.

III. A and S have a common fixed point provided that pair (S, A) is weakly compatible.

IV. B and T have a common fixed point provided pair (T, B) is weakly compatible.

V. A, B, S and T have unique common fixed point provided that (III) and (IV) both are true.

Proof. Substitute f = h = I, the identity map, p = S, q = T, k = A, g = B and φ (t) = 1 in theorem 3.1 and use the fact that every metric space is a symmetric space.

Remark 3.3. Corollary 3.3 is a improved form of the main result of Aliouche and Djoudi [3].

Corollary 3.4. Let S and T be selfmaps of a metric space (X, d).
It one of SX or TX is a complete subspace of X such that

(i) $TX \subset SX$,

(ii) $d(Tx, Ty) < \max\{d(Sx, Sy), \frac{d(Tx, Sx) + d(Ty, Sy)}{2}, \frac{d(Ty,Sx) + d(Tx,Sy)}{2}\}$

for all $x \neq y \in X$,

(iii) Pair (T, S) satisfies the property (E.A).

Then :

(I) T and S have a coincidence.

(II) T and S have a unique common fixed point provided that they are weekly compatible.

Proof. Substitute φ (t) = l, g = k = I, the identify map, p = q = T, h = f = S in theorem 3.1 and define $F : R^6_+ \to R$ by

$$F(x_1\ x_2, x_3, x_4, x_5, x_6) = x_1 - h \max\{x_2, \frac{x3 + x4}{2}, \frac{x5 + x6}{2}\}, \text{ where } 0 < h < 1.$$

Remark 3.4. Corollary 3.4 is the main result of Aamri and Moutawakil [1].

Corollary 3.5. Let A, B, S and T be selfmaps of symmetric (semi -metric) space (X, d) which satisfy properties (W.3) and (HE). If one of SX or TX is a d-closed subset of X such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) $d(Ax, By) < \max\{d(Sx, Ty), \min\{d(Ax, Sx), d(By, Ty)\}, \min\{d(Ax, Ty),$

$d(By, Sx)\}\}$.

(iii) (A, S) or (B, T) satisfies property (E.A).

Then :

(I) A and S have a coincidence.

(II) B and T have a coincidence.

(III) A and S have a common fixed point provided that they are weakly compatible.

(IV) B and T have a common fixed point provided they are weakly compatible.

(V) A, B, S and T have unique common fixed point provided that (III) and both are true.

Proof. Substitute $\varphi(t) = 1$, $g = k = I$, the identify map, $p = A$, $q = B$, $h = S$, $f = T$ and define $F : R_+^6 \to R$ by $F(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - h \max\{x_2, \min(x_3, x_4), \min(x_5, x_6)\}$, where $0 < h < 1$, in theorem 3.1.

Remark 3.5. Corollary 3.5 in the theorem 3.2 Cho et al. [6].

Corollary 3.6. Let f, g, h, k, p and q be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W.3) and (HE.). If one of p(X), q(X), hk(X) or fg(X) is a complete subspace of X such that

(i) $p(X) \subset fg(X)$, $q(X) \subset hk(X)$,

(ii) $d(px, qy) \leq h \max\{(hkx, fgy), d(px, hkx), d(qy, fgy), \frac{d(px,\ fgy)}{2}, \frac{d(hkx,\ qy)}{2}\}$, for all $x, y \in X$ and $0 < h < 1$,

(iii) (p, hk) or (q, fg) satisfies property (E. A).

Then :

(I) p and hk have a coincidence.

(II) q and fg have a coincidence.

(III) p and hk have a common fixed point provided that they are weakly compatible.

(IV) q and fg have a common fixed point provided that they are weakly compatible.

(V) p, q, hk, fg have a common fixed point provided that (III) and (IV) both are true.

(VI) f, g, h, k, p and q have a unique common fixed point provided that k commutes with p and h and g commutes with f and q.

Proof. Substitute $\varphi(t) = 1$ and define $F: R_+^6 \to R$ by $F(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - h \max\{x_2, x_3, x_4, \frac{x_5}{2}, \frac{x_6}{2}\}$ in theorem 3.1.

Corollary 3.7. Let f, g, h, k, p and q be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W.3) and (HE.). If one of p(X), q(X), hk(X) or fg(X) is a complete subspace of X such that

(i) (i)$p(X) \subset fg(X)$, $q(X) \subset hk(X)$,

(ii) $d(px, qy) \leq h \max\{(hkx, fgy), d(px, hkx), d(qy, fgy), \frac{d(px, fgy) + d(hkx, qy)}{2}\}$, for all $x, y \in X$ and $0 < h < l$,

(iii) (p, hk) or (q, fg) satisfies property (E. A).

Then :

(I) p and hk have a coincidence.

(II) q and fg have a coincidence.

(III) p and hk have a common fixed point provided that they are weakly compatible.

(IV) q and fg have a common fixed point provided that they are weakly compatible.

(V) p, q, hk and fg have a common fixed point provided that (III) and (IV) both are true.

(VI) f, g, h, k, p and q have a unique common fixed point provided that k commutes with p and h and g commutes with f and q.

Proof. Substitute $\varphi(t) = 1$ and define $F : R_+^6 \rightarrow R$ by

$F(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 - h \max \{x_2, x_3, x_4, \frac{x_5 + x_6}{2}\}$.

.**Corollary 3.8**. Let A, B, S and T be self maps of symmetric (semi-metric) space (X, d) which satisfied properties (W.3) and (HE). If one of AX, BX, SX or TX is a complete subspace of X such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) $d(Ax, By) < h \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(Sx, By)\} \leq 0$, for all $x, y \in X$, where $o < h < 1$,

(iii) (A, S) or (B, T) satisfies property (E.A).

Then :

- (I) A and S have a coincidence.
- (II) B and T have a coincidence.
- (III) A and S have a common fixed point provided that they are weakly compatible.
- (IV) B and T have a common fixed point provided they are weakly compatible.
- (V) A, B, S and T have unique common fixed point provided that (III) and (IV) both are true.

Proof : Substitute $\varphi(t) = 1$, $g = k = I$, the identity map, $p = A$, $q = B$, $h = S$, $f= T$ and define $F : R_+^6 \rightarrow R$ by $F(x_1, x_2, x_3, x_4, x_5, x_6) := x_1 - h \max\{x_2, x_3, x_4, x_5, x_6\}$ in theorem 3.1.

Corollary 3.9. Let A and B be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W.3) and (HE). If one of AX or BX is complete subspace of X such that

(i) $\int_0^{F(d(Ax,\ By),\ d(x,\ y),\ d(Ax,\ x),\ d(By,\ y),\ d(Ax,\ y),\ d(x,\ By))} \varphi(t)dt \le 0$, for all x, y $\in$ X, where F satisfy properties (F_a), (F_b) and (F_1), $\varphi : R_+ \to R$ is a Lebesgue integrable map which is summable.

(ii) (A, I) or (B, I) satisfy property (E.A).

Then :

(I) A and B have a coincidence

(II) A and B have a unique common fixed point provided that they are weakly compatible.

Proof. Substitute φ (t) = 1, f = g = h = k = I, the identity map, p = A and q = B in theorem 3.1.

Corollary 3.10. Let A and B be selfmaps of symmetric (semi-metric) space (X, d) which satisfy properties (W.3) and (HE). If one of AX or BX is complete subspace of X such that

(i) F(d(Ax, By),d(x, y), d(Ax, x), d(By, y), d(Ax, y), d(x, By)) $\le$ 0, for all x, y $\in$ X, where F $\in$ F_6 satisfy properties (F_a), (F_b) and (F_1), $\varphi : R_+ \to R$ is a Lebesgue integrable map which is summable.

(ii) (A, I) or (B, I) satisfy property (E.A)

Then :

(I) A and B have a coincidence

(II) A and B have a unique common fixed point provided that they are weakly compatible.

Proof. Substitute φ (t) = 1 in corollary 3.9.

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