先生によると、
(絶対値つけても つけなくても どちらでもいい)

問題1

[1] $\displaystyle\int \frac{dx}{x^2+1} = \mathrm{Tan}^{-1}x$

[2] $\displaystyle f(x) = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$, $\displaystyle\int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \log x - \log(x+1)$

[3] $\displaystyle f(x) = \frac{(x^2+x)'}{x^2+x}$, $\displaystyle\int \frac{(x^2+x)'}{x^2+x}dx = \log(x^2+x) = \log x(x+1) = \log x + \log(x+1)$

~~または. $\displaystyle f(x) = \frac{1}{x} + \frac{1}{x+1}$ $\displaystyle\int\left(\frac{1}{x} + \frac{1}{x+1}\right)dx = \log x + \log(x+1)$~~

[4] $\displaystyle f(x) = \frac{x}{(3x+1)(2x+1)} = \frac{A}{3x+1} + \frac{B}{2x+1}$ ← 部分分数分解

$A(2x+1) + B(3x+1) = x$ $\quad \begin{cases} 2A+3B = 1 \\ A+B = 0 \end{cases}$ $\quad \begin{matrix} A = -1 \\ B = 1 \end{matrix}$

$\displaystyle\int\left(\frac{1}{2x+1} - \frac{1}{3x+1}\right)dx = \frac{1}{2}\log(2x+1) - \frac{1}{3}\log(3x+1)$

[5] $\displaystyle f(x) = \frac{x^2-1+2}{x^2-1} = 1 + \frac{2}{x^2-1} = 1 + \frac{2}{(x-1)(x+1)} = 1 + \frac{1}{x-1} - \frac{1}{x+1}$

$\displaystyle\int\left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right)dx = x + \log(x-1) - \log(x+1)$

[6] $\displaystyle f(x) = \frac{3x^2+2x-1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$A(x^2+1) + (Bx+C)(x-2) = 3x^2+2x-1$

$(A+B)x^2 + (-2B+C)x + A - 2C = 3x^2+2x-1$ $\quad \begin{cases} A+B = 3 \\ -2B+C = 2 \\ A-2C = -1 \end{cases}$ $\quad \begin{matrix} A = 3 \\ B = 0 \\ C = 2 \end{matrix}$

$\displaystyle\int\left(\frac{3}{x-2} + \frac{2}{x^2+1}\right)dx = 3\log(x-2) + 2\,\mathrm{Tan}^{-1}x$

[7] $\displaystyle f(x) = \frac{2x^2-x+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$(Ax+B)(x^2+1) + (Cx+D) = 2x^2-x+2$ $\quad \begin{matrix} A = 0 \\ B = 2 \end{matrix}$

$Ax^3 + Bx^2 + (A+C)x + B+D = 2x^2-x+2$ $\quad \begin{matrix} A+C = -1 & C = -1 \\ B+D = 2 & D = 0 \end{matrix}$

$\displaystyle\int\left(\frac{2}{x^2+1} - \frac{x}{(x^2+1)^2}\right)dx = 2\,\mathrm{Tan}^{-1}x - \int \frac{\frac{1}{2}(x^2+1)'}{(x^2+1)^2}dx$ $\quad u = x^2+1$ とおく.

$\displaystyle = 2\,\mathrm{Tan}^{-1}x - \frac{1}{2}\int\frac{du}{u^2}$

$\displaystyle = 2\,\mathrm{Tan}^{-1}x - \frac{1}{2}\cdot\left(-\frac{1}{u}\right)$

$\displaystyle = 2\,\mathrm{Tan}^{-1}x + \frac{1}{2(x^2+1)}$

[8] $f(x) = \dfrac{1}{(x+1)(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x^2+1}$

$A(x+1)(x^2+1) + B(x-1)(x^2+1) + C(x-1)(x+1) = 1$

$(A+B)x^3 + (A-B+C)x^2 + (A+B)x + A-B-C = 1$

$\begin{cases} A+B=0 \\ A-B+C=0 \\ A+B=0 \\ A-B-C=1 \end{cases}$ $\quad A=\frac{1}{4}, \quad B=-\frac{1}{4}, \quad C=-\frac{1}{2}$

$\displaystyle \int \left\{ \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)} \right\} dx$

$\displaystyle = \frac{1}{4}\log(x-1) - \frac{1}{4}\log(x+1) - \frac{1}{2}\mathrm{Tan}^{-1}x$

問題 2

[1] $\displaystyle \int_{\substack{-1\le x\le 1 \\ -1\le y\le 1}} (2xy+4x-y-2)\,dxdy = \int_{x=-1}^{x=1} \left\{ \int_{y=-1}^{y=1} (2xy+4x-y-2)\,dy \right\} dx$

$\displaystyle = \int_{x=-1}^{x=1} \left[xy^2 + 4xy - \frac{1}{2}y^2 - 2y \right]_{y=-1}^{y=1} dx = \int_{x=-1}^{x=1} (8x-4)\,dx = \left[4x^2-4x\right]_{-1}^{1} = -8$

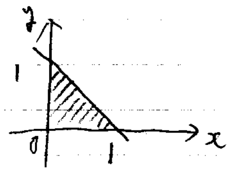
または $\displaystyle \int_{\substack{-1\le x\le 1 \\ -1\le y\le 1}} (2x-1)(y-2)\,dxdy = \int_{x=-1}^{x=1} (2x-1)\,dx \cdot \int_{y=-1}^{y=1} (y-2)\,dy = \left[x^2-x\right]_{-1}^{1} \cdot \left[\frac{1}{2}y^2-2y\right]_{-1}^{1}$

$= -8$

[2] $\displaystyle \int_{\substack{0\le x\le \pi \\ 0\le y\le \frac{\pi}{2}}} \cos(x+y)\,dxdy = \int_{x=0}^{x=\pi} \left\{ \int_{y=0}^{y=\frac{\pi}{2}} \cos(x+y)\,dy \right\} dx = \int_{x=0}^{x=\pi} \left[\sin(x+y) \right]_{y=0}^{y=\frac{\pi}{2}} dx$

$\displaystyle = \int_{x=0}^{x=\pi} \left(\sin\left(x+\frac{\pi}{2}\right) - \sin x \right) dx = \int_0^{\pi} (\cos x - \sin x)\,dx = \left[\sin x + \cos x\right]_0^{\pi} = -2$

[3] $\begin{cases} x\ge 0,\ y\ge 0 \\ y\le 1-x \end{cases}$ の領域は



$\rightarrow \begin{cases} 0\le x\le 1 \\ 0\le y\le 1-x \end{cases}$

$\displaystyle \int_{\substack{x\ge 0,\, y\ge 0 \\ x+y\le 1}} (x^2+y^2)\,dxdy = \int_{x=0}^{x=1} \left\{ \int_{y=0}^{y=1-x} (x^2+y^2)\,dy \right\} dx = \int_{x=0}^{x=1} \left[x^2y + \frac{1}{3}y^3 \right]_{y=0}^{y=1-x} dx$

$\displaystyle = \int_0^1 \left(-\frac{4}{3}x^3 + 2x^2 - x + \frac{1}{3} \right) dx = \left[-\frac{1}{3}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x \right]_0^1 = \frac{1}{6}$

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x から積分する場合、積分領域を $\begin{cases} 0\le x\le 1-y \\ 0\le y\le 1 \end{cases}$ として

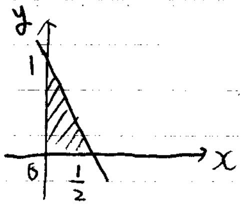
$\displaystyle \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \Leftrightarrow \int_{y=0}^{y=1} \int_{x=0}^{x=1-y}$ とする。どちらでも同じ答えとなる。

(注 xから積分する場合 $\begin{cases} 0 \le x \le \frac{1-y}{2} \\ 0 \le y \le 1 \end{cases}$)

こっちの方が計算がタイヘンかも…

問題2

[4] $\begin{cases} x \ge 0,\ y \ge 0 \\ 2x + y \le 1 \end{cases}$

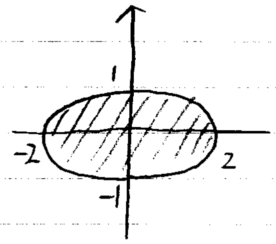


$\Rightarrow \begin{cases} 0 \le x \le \frac{1}{2} \\ 0 \le y \le 1-2x \end{cases}$

$$\int_{\substack{x\ge0, y\ge0 \\ 2x+y\le1}} e^{2x+y}\,dxdy = \int_{x=0}^{x=\frac{1}{2}} \left\{ \int_{y=0}^{y=1-2x} e^{2x+y}\,dy \right\} dx = \int_{x=0}^{x=\frac{1}{2}} e^{2x} \left[e^{y} \right]_{y=0}^{y=1-2x} dx$$

$$= \int_{x=0}^{x=\frac{1}{2}} e^{2x}(e^{1-2x}-1)\,dx = \left[ex - \frac{1}{2}e^{2x} \right]_0^{\frac{1}{2}} = \frac{1}{2}$$

[5] $\frac{x^2}{4} + y^2 \le 1$



$\begin{cases} x = 2r\cos\theta \\ y = r\sin\theta \end{cases}$ とすると、$\begin{cases} -2 \le x \le 2 \\ -1 \le y \le 1 \end{cases}$ より $\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$

ヤコビアン $J(r,\theta) = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

ヤコビアンによる変数変換により、

$= 2r\cos^2\theta + 2r\sin^2\theta = 2r$

$$\int_{x^2+4y^2\le4} (x^2+2y^2)\,dxdy = \int_{\substack{0\le r\le1 \\ 0\le\theta\le2\pi}} \left\{ (2r\cos\theta)^2 + 2(r\sin\theta)^2 \right\} \cdot \left| J(r,\theta) \right| drd\theta$$

$$= \int_{r=0}^{r=1} \left\{ \int_{\theta=0}^{\theta=2\pi} (4r^2\cos^2\theta + 2r^2\sin^2\theta)\cdot 2r\,d\theta \right\} dr = \int_{r=0}^{r=1} \left\{ \int_{\theta=0}^{\theta=2\pi} 4r^3(2\cos^2\theta + \sin^2\theta)\,d\theta \right\} dr$$

$$= \int_{r=0}^{r=1} 4r^3\,dr \cdot \int_{\theta=0}^{\theta=2\pi} \left(\frac{1+\cos2\theta}{2} + 1 \right) d\theta = \left[r^4 \right]_0^1 \cdot \left[\frac{3}{2}\theta + \frac{1}{4}\sin2\theta \right]_0^{2\pi} = 3\pi$$

(数) P.196 定理 (変数変換の公式)

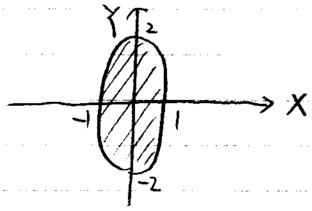
絶対値であることに注意!!

$$\iint_D f(x,y)\,dxdy = \iint_\Omega f(\varphi(s,t), \psi(s,t)) \cdot \left| J(s,t) \right| dsdt$$

問題2

[6] $(x-y)^2 + \dfrac{(x+y)^2}{4} \le 1$ $\quad \begin{cases} x-y = r\cos\theta \\ x+y = 2r\sin\theta \end{cases}$ とする.

$\begin{matrix} x-y=X \\ x+y=Y \end{matrix}$ で XY平面を考えて,



$\begin{cases} -1 \le X \le 1 \\ -2 \le Y \le 2 \end{cases}$ より $\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$

$$\begin{cases} x = \dfrac{r\cos\theta + 2r\sin\theta}{2} \\[2ex] y = \dfrac{2r\sin\theta - r\cos\theta}{2} \end{cases}$$

ヤコビアン $J(r,\theta) = \begin{vmatrix} \dfrac{\cos\theta + 2\sin\theta}{2} & \dfrac{-r\sin\theta + 2r\cos\theta}{2} \\[2ex] \dfrac{2\sin\theta - \cos\theta}{2} & \dfrac{2r\cos\theta + r\sin\theta}{2} \end{vmatrix}$

$$= \frac{r}{4}\left\{(\cos\theta + 2\sin\theta)(2\cos\theta + \sin\theta) - (-\sin\theta + 2\cos\theta)(2\sin\theta - \cos\theta)\right\}$$

$$= r$$

$$\int_{4(x-y)^2 + (x+y)^2 \le 4} xy \, dx dy = \int_{\substack{0 \le r \le 1 \\ 0 \le \theta \le 2\pi}} \left(\frac{r\cos\theta + 2r\sin\theta}{2}\right)\left(\frac{2r\sin\theta - r\cos\theta}{2}\right) \cdot r \, dr d\theta$$

$$= \frac{1}{4}\int_{r=0}^{r=1} r^3 \, dr \int_{\theta=0}^{\theta=2\pi} (\cos\theta + 2\sin\theta)(2\sin\theta - \cos\theta) \, d\theta = \frac{1}{4}\left[\frac{1}{4}r^4\right]_0^1 \cdot \int_0^{2\pi} (4\sin^2\theta - \cos^2\theta) d\theta$$

$$= \frac{1}{16}\left[\frac{3}{2}\theta - \frac{5}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{16}\pi$$

問題3

[1] $\int_0^\infty xe^{-2x}dx = \lim_{\alpha\to\infty}\int_0^\alpha xe^{-2x}dx = \lim_{\alpha\to\infty}\left\{\left[-\frac{1}{2}xe^{-2x}\right]_0^\alpha + \frac{1}{2}\int_0^\alpha e^{-2x}dx\right\}$

$= \lim_{\alpha\to\infty}\left(-\frac{1}{2}\alpha e^{-2\alpha}\right) + \frac{1}{2}\lim_{\alpha\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_0^\alpha = -\frac{1}{4}\lim_{\alpha\to\infty}(e^{-2\alpha}-1) = \frac{1}{4}$

[2] $\int_1^\infty \frac{\log x}{x^2}dx = \lim_{\alpha\to\infty}\int_1^\alpha \frac{\log x}{x^2}dx = \lim_{\alpha\to\infty}\left\{\left[-\frac{1}{x}\log x\right]_1^\alpha + \int_1^\alpha \frac{1}{x^2}dx\right\}$

$= \lim_{\alpha\to\infty}\left(-\frac{1}{\alpha}\log\alpha\right) + \lim_{\alpha\to\infty}\left[-\frac{1}{x}\right]_1^\alpha = \lim_{\alpha\to\infty}\left(-\frac{1}{\alpha}+1\right) = 1$

[3] $x = 2\sin\theta$ とすると、 $dx = 2\cos\theta\, d\theta$

x	$0 \to 2$
θ	$0 \to \frac{\pi}{2}$

$\int_0^2 \frac{1}{\sqrt{4-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{2\cos\theta}d\theta = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$

または、 $x = 2y$ として $dx = 2dy$

x	$0 \to 2$
y	$0 \to 1$

$\int_0^2 \frac{1}{\sqrt{4-x^2}}dx = \int_0^1 \frac{2dy}{2\sqrt{1-y^2}} = \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \lim_{\alpha\to 1-0}\int_0^\alpha \frac{dy}{\sqrt{1-y^2}} = \lim_{\alpha\to 1-0}\left[\mathrm{Sin}^{-1}y\right]_0^\alpha$

$= \lim_{\alpha\to 1-0}(\mathrm{Sin}^{-1}\alpha) = \frac{\pi}{2}$

[4] $\frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}}$. $e^x = y$ とすると、 $dy = e^x dx$ 、 $dx = \frac{dy}{y}$

x	$-\infty \to \infty$
y	$0 \to \infty$

$\int_{-\infty}^\infty \frac{1}{\cosh x}dx = \int_0^\infty \frac{2}{y+y^{-1}}\cdot\frac{dy}{y} = \int_0^\infty \frac{2}{y^2+1}dy = \lim_{\alpha\to\infty}\int_0^\alpha \frac{2}{y^2+1}dy$

$= \lim_{\alpha\to\infty} 2\left[\mathrm{Tan}^{-1}y\right]_0^\alpha = 2\cdot\lim_{\alpha\to\infty}(\mathrm{Tan}^{-1}\alpha) = 2\cdot\frac{\pi}{2} = \pi$

逆関数 と 双曲線関数 (cosh 、sinh)

$\int\frac{dx}{1+x^2} = \mathrm{Tan}^{-1}x$, $\int\frac{dx}{\sqrt{1-x^2}} = \mathrm{Sin}^{-1}x$

$\sinh x = \frac{e^x - e^x}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$

(注) これだけ、解法を知らないと解きにくい問題です！

問題3

[5] $\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy\right) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)}dxdy$ により、

$\int_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)}dxdy$ を求めて、$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx$ を求める。

$\mathbb{R}^2 \rightarrow \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \end{cases}$ $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ とすると、$\begin{cases} 0 \leq r < \infty \\ 0 \leq \theta \leq 2\pi \end{cases}$、$J(r,\theta) = r$

$$\int_{\mathbb{R}^2} e^{-\frac{1}{2}(x^2+y^2)}dxdy = \int_{\substack{0 \leq r < \infty \\ 0 \leq \theta \leq 2\pi}} e^{-\frac{1}{2}(r^2\cos^2\theta + r^2\sin^2\theta)} \cdot r \, drd\theta$$

$$= \int_{r=0}^{r=\infty} re^{-\frac{1}{2}r^2}dr \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi\int_0^{\infty} re^{-\frac{1}{2}r^2}dr = 2\pi\int_0^{\infty} e^{-\frac{1}{2}r^2}\cdot\left(\frac{1}{2}r^2\right)'dr$$

$$= 2\pi \lim_{\alpha\to\infty}\int_0^{\alpha} e^{-u}du \quad \left(u = \frac{1}{2}r^2 \quad \begin{array}{c|c} r & 0\to\infty \\ \hline u & 0\to\infty \end{array}\right)$$

$$= 2\pi \lim_{\alpha\to\infty}\left[-e^{-u}\right]_0^{\alpha} = 2\pi\lim_{\alpha\to\infty}(-e^{-\alpha}+1) = 2\pi$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx = \sqrt{2\pi}$$

問題3

[6] $\begin{cases} x = r\cos\theta \\ y = \frac{1}{2}r\sin\theta \end{cases}$ とする。 $\mathbb{R}^2 \to \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \end{cases}$ より $\begin{cases} 0 \leq r < \infty \\ 0 \leq \theta \leq 2\pi \end{cases}$,

ヤコビアン $J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \frac{1}{2}\sin\theta & \frac{1}{2}r\cos\theta \end{vmatrix} = \frac{1}{2}r\cos^2\theta + \frac{1}{2}r\sin^2\theta = \frac{1}{2}r$

$$\int_{\mathbb{R}^2} \frac{1}{(x^2+4y^2+1)^2}\,dxdy = \int_{\substack{0 \leq r < \infty \\ 0 \leq \theta \leq 2\pi}} \frac{1}{(r^2\cos^2\theta + r^2\sin^2\theta + 1)^2} \cdot \frac{1}{2}r\,drd\theta$$

$$= \frac{1}{2}\int_{r=0}^{r=\infty} \frac{r}{(r^2+1)^2}\,dr \cdot \int_{\theta=0}^{\theta=2\pi} d\theta = \pi \int_0^\infty \frac{\frac{1}{2}(r^2+1)'}{(r^2+1)^2}\,dr$$

$$= \frac{\pi}{2}\int_1^\infty \frac{du}{u^2} \quad \left(u = r^2+1 \text{ として, } \begin{array}{c|c} r & 0 \to \infty \\ \hline u & 1 \to \infty \end{array} \right)$$

$$= \frac{\pi}{2}\lim_{\alpha\to\infty}\int_1^\alpha \frac{du}{u^2} = \frac{\pi}{2}\lim_{\alpha\to\infty}\left[-\frac{1}{u}\right]_1^\alpha = \frac{\pi}{2}\lim_{\alpha\to\infty}\left(-\frac{1}{\alpha}+1\right) = \frac{\pi}{2}$$