熱統計学の κ-拡張とその情報幾何構造
A κ-extension of thermstatistics and the related information geometric structures

和田 達明 [1], 松添 博 [2] and Antonio M. Scarfone[3]

[1] 茨城大学 工学部 電気電子工学科
[2] 名古屋工業大学 情報工学専攻 数理情報分野
[3]Istituto dei Sistemi Complessi (ISC-CNR) - c/o Politecnico di Torino, Italy

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Outline

Generalized thermostatistics

- Thermostatistics

κ-統計力学

- Finite κ-difference and κ-averaging operators
- κ-generalized Nonlinear Fokker-Planck equations
- κ-generalized Linear Fokker-Planck equation

Information Geometry

κ-変形指数分布に基づく情報幾何構造

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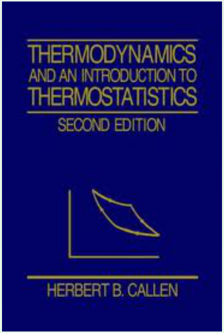
Thermostatistics (熱統計学)

H.B. Callen's book (John Wiley & Sons 1985)



From Chap. 21

Unlike mechanics, thermostatistics is not a detailed theory of dynamic response to specified forces. And unlike electromagnetic theory, thermostatistics is not a theory of the forces themselves.

Thermostatistics characterizes the equilibrium state of microscopic systems without reference either to the specific forces or to the laws of mechanical response.

Instead thermostatistics characterizes the equilibrium state as the state that maximizes the disorder, a quantity associated with a conceptual framework ("information theory")

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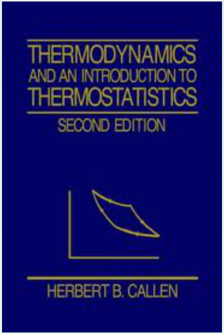

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the disorder $\Leftrightarrow$ Shannon entropy

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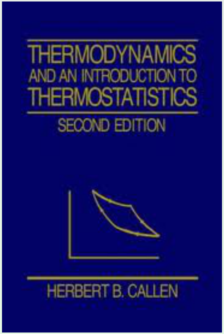

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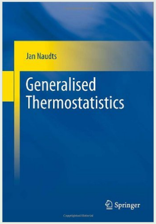
Generalized thermostatistics



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A generalization of Callen's thermostatistics based on a generalized entropy:

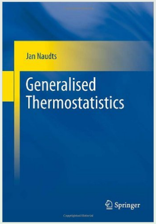
$$S_\phi = -\sum_i p_i \ln_\phi p_i \quad \underset{\phi(s)\to s}{\longrightarrow} \quad S^{\mathrm{BGS}},$$

$$\text{with} \quad \ln_\phi x \equiv \int_1^x \frac{ds}{\phi(s)}, \quad \underset{\phi(x)\to s}{\longrightarrow} \quad \ln x.$$

Examples:

- Tsallis: $\phi(s) = s^q, \quad \ln_q(x) = \frac{x^{1-q}-1}{1-q}, \quad q > 0$
- Kaniadakis:
  $\phi(s) = 2s/(s^\kappa + s^{-\kappa}), \quad \ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \quad -1 < \kappa < 1$

Generalized thermostatistics



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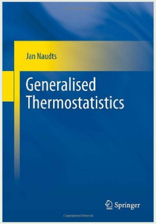

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κ-エントロピー

Gibbs-Shannon エントロピーの 1 パラメータ（κ) 拡張

$$S_\kappa \equiv -\int dx\, p(x) \ln_\kappa p(x) \underset{\kappa\to 0}{\longrightarrow} \quad S^{\mathrm{GS}} = -\int dx\, p(x) \ln p(x),$$

$$\kappa\text{-対数関数：}\ \ln_\kappa(x) \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \quad \underset{\kappa\to 0}{\longrightarrow} \quad \ln(x).$$

by G. Kaniadakis and A.M. Scarfone, Politecnico di Torino, Italy.

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κ-指数関数： $$\exp_\kappa(x) \equiv \left(\kappa\, x + \sqrt{1+\kappa^2\, x^2}\right)^{1/\kappa}, \quad \xrightarrow[\kappa\to 0]{} \quad \exp(x)$$

G. Kaniadakis, Physica A **296**, 405 (2001).

G. Kaniadakis, Phys. Rev. E **66**, 056125 (2002).

G. Kaniadakis, A.M. Scarfone, Physica A **305**, 69 (2002).

G. Kaniadakis, Phys. Rev. E **72** 036108 (2005).

$\ln_\kappa(x)$ and $u_\kappa(x)$

κ-ユニット関数：

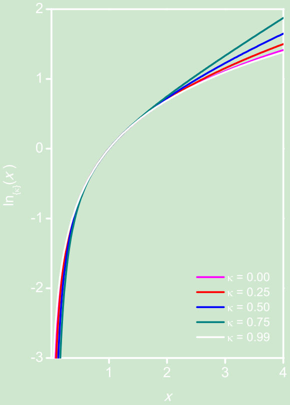
$$u_\kappa(x) \equiv \frac{x^\kappa + x^{-\kappa}}{2}\,, \qquad \xrightarrow[\kappa\to 0]{} \qquad 1$$

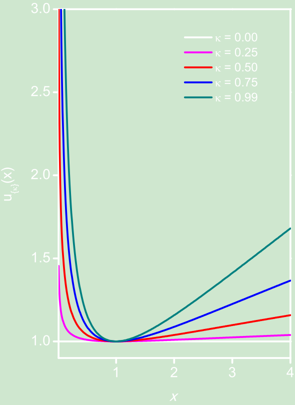
Similar to the κ-entropy, we introduce the new function $\mathcal{I}_\kappa$ as the average of the $u_\kappa(x)$:

$$S_\kappa[p] = -\langle \ln_\kappa[p] \rangle \equiv -\sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\,\kappa}\,,$$

$$\mathcal{I}_\kappa[p] = \langle u_\kappa[p] \rangle \equiv \sum_i \frac{p_i^{1+\kappa} + p_i^{1-\kappa}}{2} \qquad \xrightarrow[\kappa\to 0]{} \qquad \sum_i p_i = 1\,.$$

$\ln_\kappa(x)$ and $u_\kappa(x)$





- $\ln_\kappa(\Re^+) \in \Re$
- $\ln_\kappa(1) = 0$
- $\ln_\kappa(1/x) = -\ln_\kappa(x)$
- $\ln_\kappa(1/\alpha) = 1/\lambda$

- $u_\kappa(\Re^+) \in \Re^+$
- $u_\kappa(1) = 1$
- $u_\kappa(1/x) = u_\kappa(x)$
- $u_\kappa(1/\alpha) = 1/\lambda$

$\ln_\kappa(x)$ and $u_\kappa(x)$

A relevant representation of $\ln_\kappa(x)$ and $u_\kappa(x)$ is given by

$$\ln_\kappa(x) = \frac{1}{\kappa}\sinh\big(\kappa \ln(x)\big), \qquad u_\kappa(x) = \cosh\big(\kappa \ln(x)\big)$$

Thus we have the following relations:

$$u_\kappa^2(x) - \kappa^2 \ln_\kappa^2(x) = 1 \,,$$

$$u_\kappa(x^2) = u_\kappa^2(x) + \kappa^2 \ln_\kappa^2(x) \,, \qquad \ln_\kappa(x^2) = 2 \ln_\kappa(x)\, u_\kappa(x) \,,$$

$$\frac{d}{dx}\ln_\kappa(x) = \frac{1}{x}\, u_\kappa(x) \,, \qquad \frac{d}{dx} u_\kappa(x) = \frac{\kappa^2}{x} \ln_\kappa(x) \,.$$

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Outline

Finite κ-difference and κ-averaging operators

Difference operator:

$$D_\epsilon f(x) \equiv \frac{f(x+\epsilon) - f(x-\epsilon)}{2\,\epsilon}\,, \qquad \underset{\epsilon\to 0}{\longrightarrow} \qquad \frac{d}{dx} f(x),$$

Averaging operator:

$$M_\epsilon f(x) \equiv \frac{f(x+\epsilon) + f(x-\epsilon)}{2}\,, \qquad \underset{\epsilon\to 0}{\longrightarrow} \qquad f(x)$$

that fulfill Leibniz rule:

$$D_\epsilon\,[f(x)\,g(x)] = D_\epsilon\,f(x)\,M_\epsilon\,g(x) + M_\epsilon\,f(x)\,D_\epsilon\,g(x)\,,$$
$$M_\epsilon\,[f(x)\,g(x)] = M_\epsilon\,f(x)\,M_\epsilon\,g(x) + \epsilon^2\,D_\epsilon\,f(x)\,D_\epsilon\,g(x)\,.$$

Finite κ-difference and κ-averaging operators

By introducing the generators

$$g^x(s) = x^s \,, \qquad \text{and} \qquad G^{\{x_\mu\}}(s) = \sum_\mu p_\mu \, x_\mu^s \,,$$

we get

$$\ln_\kappa(x) = D_\kappa \, g^x(s)\Big|_{s=0} \,, \qquad S_\kappa[p] = -D_\kappa \, G^{\{p_\mu\}}(s)\Big|_{s=0} \,,$$

$$u_\kappa(x) = M_\kappa \, g^x(s)\Big|_{s=0} \,, \qquad \mathcal{I}_\kappa[p] = M_\kappa \, G^{\{p_\mu\}}(s)\Big|_{s=0} \,,$$

Composability

From the addition rules for sinh(x) and cosh(x) we can obtain

$$\ln_\kappa(x\,y) = \ln_\kappa(x)\,u_\kappa(y) + u_\kappa(x)\,\ln_\kappa(y)\;,$$
$$u_\kappa(x\,y) = u_\kappa(x)\,u_\kappa(y) + \kappa^2\,\ln_\kappa(x)\,\ln_\kappa(y)\;,$$

so that, for independent statistical systems, with $p^{AB}_{\mu\nu} = p^A_\mu \cdot p^B_\nu$

$$S_\kappa[p^{AB}] = S_\kappa[p^A]\,\mathcal{I}_\kappa[p^B] + \mathcal{I}_\kappa[p^A]\,S_\kappa[p^B]\;,$$
$$\mathcal{I}_\kappa[p^{AB}] = \mathcal{I}_\kappa[p^A]\,\mathcal{I}_\kappa[p^B] + \kappa^2\,S_\kappa[p^A]\,S_\kappa[p^B]\;.$$

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κ-MaxEnt

$$\frac{\delta}{\delta p}\Big(S_\kappa[p] - \beta U[p] - \gamma \int p(x)dx\Big) = 0.$$

$$\text{with} \quad U[p] = \int_{-\infty}^{\infty} dx\, \frac{x^2}{2} p(x),$$

leads to κ-Gaussian:

$$p_{\mathrm{ME}}(x) = \alpha \,\exp_\kappa\left[-\frac{1}{\lambda}(\gamma + \beta\,\frac{x^2}{2})\right],$$

where α and λ are κ-dependent constants,

$$\alpha = \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{1}{2\kappa}}, \quad \lambda = \sqrt{1-\kappa^2}.$$

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κ-generalized Fokker-Planck equation

T. Wada, A.M. Scarfone, AIP Conf. Proc. **965** (2007) 177.

κ-generalized NLFPE:

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\Big(x\, p(x,t)\Big) + D \frac{\partial^2}{\partial x^2}\left(\frac{p(x,t)^{1+\kappa} + p(x,t)^{1-\kappa}}{2}\right),$$

where D is a constant diffusion coefficient.

$$\underset{\kappa \to 0}{\longrightarrow} \quad \frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\Big(x\, p(x,t)\Big) + D \frac{\partial^2}{\partial x^2} p(x,t).$$

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Minimum κ-generalized free energy

The Lyapunov functional for the κ-NLFPE:
$\mathcal{L}_\kappa(t) \equiv U[p] - D\, S_\kappa[p]$ is non-increase, i.e., $\frac{d}{dt}\mathcal{L}_\kappa(t) \leq 0$

Consequently $\mathcal{L}_\kappa(t)$ is minimized for the stationary solution:

$$\min \mathcal{L}_\kappa(t) = \lim_{t\to\infty} \mathcal{L}_\kappa(t) = U[p_{\mathrm{ME}}] - D\, S_\kappa[p_{\mathrm{ME}}] = F_\kappa[p_{\mathrm{ME}}]$$

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Stationary solution of κ-NLFPE

The stationary solution $=$ the optimal MaxEnt state

$$
\begin{aligned}
p_{\mathrm{st}}(x) &\equiv \lim_{t\to\infty} p(x,t) \\
&= \alpha \exp_\kappa \left[-\frac{1}{\lambda}\left(\gamma + \frac{1}{D}\frac{x^2}{2}\right)\right] = p_{\mathrm{ME}}(x)
\end{aligned}
$$

κ-Gaussian!

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κ-Gaussian!

κ-generalized diffusion equation

κ-NLFP equation:

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial}{\partial x}\Big(x\,p(x,t)\Big) + D\frac{\partial^2}{\partial x^2}\left(\frac{p(x,t)^{1+\kappa} + p(x,t)^{1-\kappa}}{2}\right),$$

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Porous medium equation

PME($m > 1$): a NL transport equation for a fluid in porous medium

$$\frac{\partial}{\partial t}\rho(x,t) = D\,\frac{\partial^2}{\partial x^2}\,\rho^m(x,t), \quad m \in R^+,$$

$$\begin{cases} \frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\mathbf{v}) = 0 & \text{(continuity of mass)} \\ \mathbf{v} \propto -\mathrm{grad}P & \text{(Darcy's law)} \\ P \propto \rho^{\nu} & \text{(polytropic fluid)} \end{cases}$$

κ-generalized diffusion equation

$$\frac{\partial}{\partial t}\rho(x,t) = D\,\frac{\partial^2}{\partial x^2}\left(\frac{\rho^{1+\kappa}(x,t)+\rho^{1-\kappa}(x,t)}{2}\right), \quad -1<\kappa<1,$$

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Non-Gaussians with power-law tails

Frequently observed in a variety of systems. E.g.,

- Lévy stable distributions:

$$\mathcal{L}_\alpha^C(x) = \frac{1}{2\pi}\int dk \exp[ikx - C|k|^\alpha], \quad (0 < \alpha \leq 2)$$

- Tsallis' q-generalized distributions:

$$W_q(x) = \frac{1}{Z_q}\exp_q\left[-\beta x^2\right], \quad (1 < q \leq 3)$$

- Kaniadakis' κ-generalized distributions:

$$W_\kappa(x) = \frac{1}{Z_\kappa}\exp_\kappa\left[-\beta x^2\right], \quad (|\kappa| \leq 2)$$

A common key feature: an asymptotic power-law tail.

Anomalous diffusion in an optical lattice

Lutz predicted that the stationary momentum distributions of trapped atoms in an optical lattice are Tsallis' q-generalized Gaussian:

$$W_q(p) \propto \left[1 - \beta(1-q)p^2\right]^{\frac{1}{1-q}} .$$

E. Lutz, PRA **67**, 051402 (2003); PRL **93**, 190602 (2004).

Gaeta showed its invariance under the asymptotic Lie symmetries.

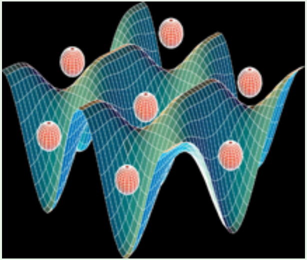
G. Gaeta, PRA **72**, 033419 (2005).

Lutz's prediction was experimentally verified by Douglas *et. al.*.

P. Douglas, *et. al.*, PRL **96**, 110601 (2006).

PHYSICAL REV

PRL **96,** 110601 (2006)



An egg-carton-like potential for atoms in an optical lattice.

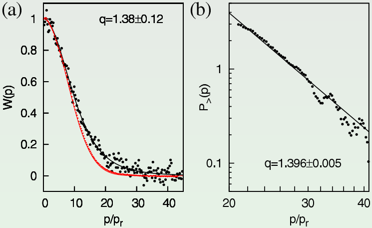


FIG. 3 (color online). (a) Experimental results for the atomic momentum distribution (black data points) and their best fit with a Tsallis function (black solid line). The value of the q parameter derived from the fit is indicated in the figure. The adjusted R^2 is equal to 0.9985. The parameters of the optical lattice are $\Delta = -24\Gamma$, $\omega_\nu = (2\pi)20.6$ kHz. For comparison, the experimental data and relative fit for a deep optical potential $[\omega_\nu = (2\pi)27.5$ kHz$]$ are also reported (gray points and line, red online). The best fit with a Tsallis distribution produces $q = 1.01 \pm 0.01$; i.e., it is a Gaussian. (b) The data points are the experimental results for the distribution $P_>(p)$ [see Eq. (3)]. The solid line represents the best fit with the power law $cp^{(3-q)/(1-q)}$ of the data for $P_>(p)$ in the shown interval $p > 20p_r$.

This anomurous transport is described by FP equation with a nonlinear drift coefficient,

$$K^{\mathrm{ol}}(p) = -\frac{\alpha p}{1 + \left(\frac{p}{p_c}\right)^2},$$

which represents a capture force with damping coefficient α and the capture momentum p_c.

Characteristic feature of this nonlinear drift

- for $|p| < p_c$, $\quad K^{\mathrm{ol}}(p) \sim -p$, i.e., Ornstein-Uhlenbeck process
- whereas for $|p| > p_c$, $\quad K^{\mathrm{ol}}(p) \sim -1/p$.

Motivations

Lutz' analysis can be applied to a wide class of systems described by FP equation with a drift coefficient decaying asymptotically as $-1/p$.

- **How can we extend this fact to a κ-generalized Gaussian?**
- **How do the microscopic parameters affect to the parameter κ and β?**

κ-exponential function

κ-exponential is a real-parameter (κ) extension of $\exp(x)$.

$$\exp_\kappa(x) \equiv \exp\left[\frac{1}{\kappa}\,\mathrm{arsinh}(\kappa x)\right] \quad \underset{\kappa\to 0}{\longrightarrow} \quad \exp(x).$$

Note that

$$\exp_\kappa(x) \underset{x\to 0}{\sim} \exp(x),$$

$$\exp_\kappa(x) \underset{x\to\pm\infty}{\sim} |2\kappa x|^{\pm\frac{1}{|\kappa|}}, \quad \text{asymptotic power-law!}.$$

Proposed nonlinear drift

T.W: Eur. Phys. J. B **73**, 287-291 (2010)

We consider a linear FP equation

$$\frac{\partial}{\partial t} w(p,t) = -\frac{\partial}{\partial p}\Big(K(p)\, w(p,t)\Big) + D\frac{\partial^2}{\partial p^2} w(p,t),$$

with the momentum-dependent drift coefficient,

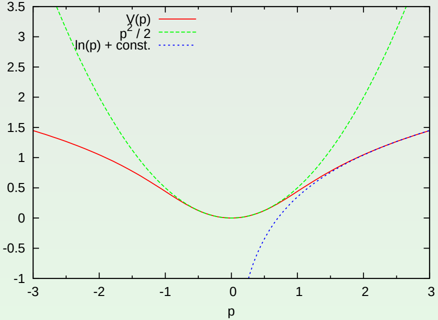
$$K(p) = -\frac{\alpha p}{\sqrt{1+\left(\frac{p}{p_c}\right)^4}}.$$

Note that $K(p)$ also asymptotically decreases as $-1/p$ for a large momentum $|p| > p_c$.

Associated potential

The potential associated with the nonlinear drift $K(p) = -\frac{d}{dp}V(p)$ is

$$V(p) = \frac{\alpha p_c^2}{2} \operatorname{arsinh}\left(\frac{p^2}{p_c^2}\right), \\ \sim \begin{cases} \frac{p^2}{2} & (p \ll p_c) \\ \ln p & (p \gg p_c) \end{cases}$$



$V(p)$ as a function of momentum p. Setting $\alpha = p_c = 1$.

Stationary state

$$\frac{\partial}{\partial t} w(p,t) = -\frac{\partial}{\partial p}\Big(K(p)\, w(p,t) + D\frac{\partial}{\partial p} w(p,t)\Big)$$

The stationary condition $\frac{\partial}{\partial t} w_s(p) = 0$ leads to

$$\frac{\partial}{\partial p} \ln w_s(p) = \frac{K(p)}{D} = -\frac{\frac{\alpha}{D} p}{\sqrt{1 + \left(\frac{p}{p_c}\right)^4}} = -\frac{\alpha p_c^2}{2D}\,\frac{\partial}{\partial p}\,\mathrm{arsinh}\left(\frac{p^2}{p_c^2}\right).$$

Stationary state

$$\frac{\partial}{\partial p} \ln w_s(p) = \frac{K(p)}{D} = -\frac{\frac{\alpha}{D} p}{\sqrt{1 + \left(\frac{p}{p_c}\right)^4}} = -\frac{\alpha p_c^2}{2D} \frac{\partial}{\partial p} \operatorname{arsinh}\left(\frac{p^2}{p_c^2}\right).$$

this becomes

$$\ln w_s(p) = \frac{\alpha p_c^2}{2D} \operatorname{arsinh}\left(-\frac{p^2}{p_c^2}\right) + \text{const.}$$

Stationary state

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$$\ln w_s(p) = \frac{\alpha p_c^2}{2D} \operatorname{arsinh}\left(-\frac{p^2}{p_c^2}\right) + \text{const.}$$

$$w_s(p) \propto \exp\left[\frac{\alpha p_c^2}{2D} \operatorname{arsinh}\left(-\frac{p^2}{p_c^2}\right)\right]$$

Stationary state

$$w_s(p) \propto \exp\left[\frac{\alpha p_c^2}{2D}\,\mathrm{arsinh}\left(-\frac{p^2}{p_c^2}\right)\right]$$

With the help of the κ-exponential and

$$\kappa = \frac{2D}{\alpha p_c^2}, \quad \beta = \frac{\alpha}{2D},$$

we found that the stationary solution is nothing but a κ-generalized Gaussian.

$$w_s(p) \propto \exp\left[\frac{1}{\kappa}\,\mathrm{arsinh}\left(-\kappa\,\beta\,p^2\right)\right] = \exp_\kappa(-\beta\,p^2).$$

Remarkably,

i) the parameter κ is expressed in terms of the microscopic parameters

$$\kappa = \frac{2D}{\alpha p_c^2} \quad \underset{p_c \to \infty}{\longrightarrow} \quad 0, \quad \text{standard Gaussian}.$$

ii) the parameter β is expressed as the ration of the friction coefficient α to the diffusion coefficient D, in analogy with the fluctuation-dissipation relation.

Lyapunov functional

Introducing the Lyapunov functional

$$\mathcal{F}[w] \equiv U[w] - D\, S^{\mathrm{BG}}[w],$$

$$S^{\mathrm{BG}}[w] = -\int dp\, w(p,t) \ln w(p,t), \quad U[w] \equiv \int dp\, V(p)\, w(p,t).$$

The time evolution of $\mathcal{F}[p]$ is non-increasing.

$$\begin{aligned}
\frac{d\mathcal{F}}{dt} &= \int dp\, \frac{\partial}{\partial w}\left[V(p)\, w + Dw \ln w\right] \frac{\partial w(p,t)}{\partial t} \\
&= \int dp\, \left[V(p) - D(\ln w + 1)\right] \frac{\partial}{\partial p}\left[-K(p) + D\frac{\partial}{\partial p}\, w\right] \\
&= -\int dp\, w \left\{-K(p) + D\,\frac{\partial}{\partial p} \ln w\right\}^2 \leq 0.
\end{aligned}$$

$\Rightarrow \quad \mathcal{F}(\tau)$ is non-increasing!

Ergodicity breaking

κ-Gaussian

in momentum space:

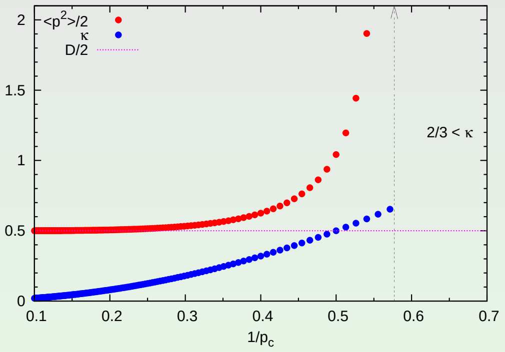
$$w_s(p) \propto \exp_\kappa\left(-\beta p^2\right), \tag{1}$$

is a κ-generalization of the standard Gaussian.

$$\text{normalization:} \quad \int_{-\infty}^{\infty} dp \, \exp_\kappa\left(-\beta p^2\right) < \infty, \quad (|\kappa| < 2)$$

$$\text{2nd moment:} \quad \int_{-\infty}^{\infty} dp \, p^2 \exp_\kappa\left(-\beta p^2\right) < \infty, \quad (|\kappa| < \frac{2}{3}).$$

$\Rightarrow$ for $2/3 \leq |\kappa| \leq 2$, the second moment is infinite, and the mean kinetic energy, $\langle p^2 \rangle / 2m$, diverges!



The variations of the second moment $\langle p^2 \rangle /2$ for the κ-Gaussian against the inverse capture momentum $1/p_c$. The other microscopic parameters are set to $D = \alpha = 1$, so $\kappa = 2/p_c^2$ and $\beta = 1/2$.

情報幾何学

甘利、長岡: *Methods of Information Geometry*, (AMS 2001)

$$\text{統計モデル}: \mathcal{S} = \left\{ p_\theta(x) \,\middle|\, p_\theta(x) > 0, \int dx\, p_\theta(x) = 1 \right\},$$

パラメータ： $\boldsymbol{\theta} = (\theta^1, \theta^2 \ldots, \theta^M)$.

$\Leftarrow$ $\mathcal{S}$ を微分多様体 $\mathcal{M}$ と見做す。
$\{\theta^i\}$ を局所座標系、 Fisher 情報行列:

$$g^{\mathrm{F}}_{ij}(\boldsymbol{\theta}) = \langle \partial_i \ell_\theta(x)\, \partial_j \ell_\theta(x) \rangle, \quad i, j = 1, 2, \ldots, M,$$

$$\text{where } \ell_\theta(x) \equiv \ln p_\theta(x)$$

を Riemannian 計量とし、 affine 接続を備えた微分多様体。
ここで、$\langle \cdot \rangle$ は、$p(x; \boldsymbol{\theta})$ に関する期待値。 $\partial_i = \partial / \partial \theta^i$.

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双対 affine 接続

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^\star_Z Y)$$

$$\nabla \quad \Leftrightarrow \quad \nabla^\star \quad \text{互いに双対な接続}$$

$$\partial_i g^{\mathrm{F}}_{jk} = \Gamma^{(e)}_{ij,k} + \Gamma^{(m)}_{ij,k},$$

e-接続と m-接続（の第 1 種 Christoffel 記号）：

$$\Gamma^{(e)}_{ij,k} \equiv \int dx\, \partial_k p_\theta(x) \partial_i \partial_j \ell_\theta(x) = \langle \partial_k \ell_\theta \; \partial_i \partial_j \ell_\theta \rangle,$$

$$\Gamma^{(m)}_{ij,k} \equiv \int dx\, \partial_i \partial_j p_\theta(x) \partial_k \ell_\theta(x) = \left\langle \frac{1}{p(x;\boldsymbol{\theta})} \partial_i \partial_j p(x;\boldsymbol{\theta}) \; \partial_k \ell_\theta \right\rangle,$$

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指数型分布

$$\mathcal{S}_{\rm exp} = \left\{ p_\theta(x) \,\middle|\, p_\theta(x) = \exp\left[\sum_{m=1}^{M} \theta^m f_m(x) - \Psi(\boldsymbol{\theta})\right], \right.$$
$$\left. \int dx\, p_\theta(x) = 1,\ \boldsymbol{\theta} \in \mathbb{R} \right\}.$$

- 統計多様体 $(\mathcal{S}_{exp}, g^{\rm F}, \nabla^{(e)}, \nabla^{(m)})$：双対平坦
- $\{\theta^i\}$: $\nabla^{(e)}$-affine 座標
- $\eta_i = \langle f_i \rangle = \int dx p_\theta(x) f_i(x), \quad \Rightarrow \quad \{\eta_i\}$: $\nabla^{(m)}$-affine 座標
- $\theta^i = \partial^i \Psi^*(\boldsymbol{\eta}), \quad \eta_i = \partial_i \Psi(\boldsymbol{\theta}),$

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- $\theta^i = \partial^i \Psi^*(\boldsymbol{\eta}), \quad \eta_i = \partial_i \Psi(\boldsymbol{\theta}),$

Fisher 計量

$$g^{\mathrm{F}}_{ij}(\boldsymbol{\theta}) = \partial_i \partial_j \Psi(\boldsymbol{\theta})$$
$$= \left\langle \left(f_i - \langle f_i \rangle\right) \left(f_j - \langle f_j \rangle\right) \right\rangle, \quad i, j = 1, 2, \ldots, M,$$

$\Rightarrow$ 統計モデル $\mathcal{S}_{\mathrm{exp}}$ に対する共分散行列.

ポテンシャル関数の Legendre 関係：

$$\Psi(\boldsymbol{\theta}) + \Psi^*(\boldsymbol{\eta}) - \boldsymbol{\theta} \cdot \boldsymbol{\eta} = 0,$$

e-平坦 (exponential-flat)

$$\Gamma^{(e)}_{ij,k} = \left\langle \partial_i \partial_j \ell_\theta(x)\, \partial_k \ell_\theta(x) \right\rangle = 0, \quad \forall\, i, j, k,$$

θ^i: e-affine 座標

m-平坦 (mixture-flat)

$$\Gamma^{(m)}_{ij,k} = \left\langle \frac{1}{p(x;\boldsymbol{\eta})} \partial^i \partial^j p(x;\boldsymbol{\eta})\, \partial^k \ln p(x;\boldsymbol{\eta}) \right\rangle = 0, \quad \forall\, i, j, k,$$

identically, where $\partial^i = \partial/\partial\eta_i$, and in this case, the set of coordinates $\boldsymbol{\eta}$ is called m-affine coordinates.

指数型分布族の例

大正準集合

$$p^{\mathrm{G}}(x) = \frac{1}{Z^{\mathrm{G}}(\beta, \mu)} \exp\left[-\beta E_N(x) + \beta \mu N\right],$$

with

$$Z^{\mathrm{G}}(\beta, \mu) = \sum_{N=0}^{\infty} \int dx \, \exp[-\beta E_N(x) + \beta \mu N].$$

ここで、N は粒子数、 $E_N(x)$ は N 粒子の配置 x でのエネルギー。

$\theta^1 = -\beta, \theta^2 = \beta\mu, f_1(x) = E_N(x), f_2(x) = N$, and

$$\Psi(\boldsymbol{\theta}) = \ln Z^{\mathrm{G}}(\beta, \mu) = \Phi(\beta, \mu), \quad \text{Massieu potential}$$

Fisher 情報行列 g^{FG} は、θ-ポテンシャル関数の Hesse 行列:

$$g^{\mathrm{FG}}_{ij} = \partial_i \partial_j \ln Z^G(\beta, \mu) = \partial_i \partial_j \Phi(\beta, \mu),$$

η-座標:

$$\eta^G_1 = -\frac{\partial}{\partial \beta} \ln Z^G(\beta, \mu) = \langle E_N \rangle,$$
$$\eta^G_2 = \frac{\partial}{\partial (\beta\mu)} \ln Z^G(\beta, \mu) = \langle N \rangle,$$

η-ポテンシャル関数:

$$\Psi^*(\boldsymbol{\eta}) = -\beta \langle E_N \rangle + \beta\mu \langle N \rangle - \ln Z_G(\beta, \mu) = -S^{\mathrm{GS}}.$$

$$g^{\mathrm{FG}} = \partial_i\, \eta_j = \begin{pmatrix} -\frac{\partial}{\partial\beta}\left\langle E_N \right\rangle & -\frac{\partial}{\partial\beta}\left\langle N \right\rangle \\ \frac{\partial}{\partial(\beta\mu)}\left\langle E_N \right\rangle & \frac{\partial}{\partial(\beta\mu)}\left\langle N \right\rangle \end{pmatrix}$$

Fisher 計量の（添字に関する）対称性 $g^{\mathrm{FG}}_{ij} = g^{\mathrm{FG}}_{ji}$：

$$g^{\mathrm{FG}}_{ij} = \partial_i \partial_j \Psi(\boldsymbol{\theta}) = \partial_j \partial_i \Psi(\boldsymbol{\theta}) = g^{\mathrm{FG}}_{ji}$$

Maxwell 関係式

$$-\frac{\partial}{\partial\beta}\left\langle N \right\rangle = \frac{\partial}{\partial(\beta\mu)}\left\langle E_N \right\rangle,$$

$$g^{\mathrm{FG}} = \partial_i \, \eta_j = \begin{pmatrix} -\frac{\partial}{\partial \beta} \langle E_N \rangle & -\frac{\partial}{\partial \beta} \langle N \rangle \\ \frac{\partial}{\partial (\beta\mu)} \langle E_N \rangle & \frac{\partial}{\partial (\beta\mu)} \langle N \rangle \end{pmatrix}$$

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$$-\frac{\partial}{\partial \beta} \langle N \rangle = \frac{\partial}{\partial (\beta\mu)} \langle E_N \rangle ,$$

揺動応答（fluctuation-response）関係

$$g^{\mathrm{FG}} = \begin{pmatrix} -\frac{\partial}{\partial\beta}\langle E_N\rangle & -\frac{\partial}{\partial\beta}\langle N\rangle \\ \frac{\partial}{\partial(\beta\mu)}\langle E_N\rangle & \frac{\partial}{\partial(\beta\mu)}\langle N\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle E_N^2\rangle - \langle E_N\rangle^2 & \langle E_N N\rangle - \langle E_N\rangle\langle N\rangle \\ \langle E_N N\rangle - \langle E_N\rangle\langle N\rangle & \langle N^2\rangle - \langle N\rangle^2 \end{pmatrix}. \quad \Leftarrow \quad e\text{-平坦}$$

熱平衡における "熱ゆらぎ "と "応答関数 (感受率) "を結びつける.

揺動応答（fluctuation-response）関係

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熱平衡における ”熱ゆらぎ ”と ”応答関数 (感受率) ”を結びつける.

κ-エントロピー:

$$S_\kappa \equiv -\int dx\, p(x) \ln_\kappa p(x) = \langle -\ln_\kappa p \rangle,$$

$$\kappa\text{-対数関数: } \ln_\kappa x \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa} = \frac{1}{\kappa} \sinh\left(\kappa \ln x\right),$$

for $x > 0$ and a real-parameter $\kappa \in (-1, 1)$.

$\ln_\kappa x$ の逆関数は、 κ-指数関数：

$$\exp_\kappa(x) \equiv \left[\kappa x + \sqrt{1 + \kappa^2 x^2}\right]^{\frac{1}{\kappa}}.$$

$$\lim_{\kappa \to 0} \exp_\kappa(x) = \exp(x),$$

$$\lim_{\kappa \to 0} \ln_\kappa x = \ln x.$$

κ-変形 ユニット関数:

$$u_\kappa(x) \equiv \frac{x^\kappa + x^{-\kappa}}{2} = \cosh\left(\kappa \ln x\right),$$

$\Leftarrow \quad \ln_\kappa x = \frac{1}{\kappa} \sinh\left(\kappa \ln x\right)$ に共役な変形関数。

$$\lim_{\kappa \to 0} u_\kappa(x) = 1.$$

κ- エントロピー S_κ が $-\ln_\kappa p(x)$ の期待値であるのと同様に:

$$\mathcal{I}_\kappa \equiv \int dx\, p(x) u_\kappa\left(p(x)\right) = \langle u_\kappa(p) \rangle,$$

$\Leftarrow \quad u_\kappa(p(x))$ の期待値.

κ- エントロピー 最大原理:

$$\max_{p(x)} \left(S_\kappa - \sum_{m=1}^{M} \theta^m \int dx\, f_m(x) p(x) - \gamma \int dx\, p(x) \right),$$

where $\{\theta^m\}$ and γ are Lagrange multipliers.

κ- エントロピー 最大原理:

$$\max_{p(x)} \left(S_\kappa - \sum_{m=1}^{M} \theta^m \int dx\, f_m(x) p(x) - \gamma \int dx\, p(x) \right),$$

where $\{\theta^m\}$ and γ are Lagrange multipliers.

To solve this, introducing two κ-dependent constants α and λ, which satisfy the condition:

$$\frac{d}{dx} \left(x \ln_\kappa x \right) = \lambda \ln_\kappa \left(\frac{x}{\alpha} \right).$$

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$$\alpha = \left(\frac{1-\kappa}{1+\kappa}\right)^{\frac{1}{2\kappa}}, \quad \lambda = \sqrt{1-\kappa^2},$$

which are related to each other according to

$$\ln_\kappa \left(\frac{1}{\alpha}\right) = \frac{1}{\lambda}.$$

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which are related to each other according to

$$\ln_\kappa\left(\frac{1}{\alpha}\right) = \frac{1}{\lambda}.$$

よって、以下の κ-変形指数型分布族を考える。

$$\mathcal{S}_{\kappa\text{-exp}} = \left\{ p_{\boldsymbol{\theta}}(x) \,\middle|\, p_{\boldsymbol{\theta}}(x) = \alpha \exp_\kappa\left[\frac{1}{\lambda}\left(\sum_{m=1}^{M}\theta^m f_m(x) - \gamma(\boldsymbol{\theta})\right)\right],\right.$$
$$\left.\int dx\, p_{\boldsymbol{\theta}}(x) = 1\right\}.$$

Legendre 関係式

$$\Psi_\kappa(\boldsymbol{\theta}) = \boldsymbol{\theta} \cdot \boldsymbol{\eta} - \Psi_\kappa^\star(\boldsymbol{\eta}),$$

θ- 及び η- ポテンシャル関数:

$$\Psi_\kappa(\boldsymbol{\theta}) = \mathcal{I}_\kappa(\boldsymbol{\theta}) + \gamma(\boldsymbol{\theta}),$$
$$\Psi_\kappa^\star(\boldsymbol{\eta}) = -S_\kappa(\boldsymbol{\eta}).$$

$\mathcal{I}_\kappa(\boldsymbol{\theta}) = \langle u_\kappa(p_\theta) \rangle$

κ-escort 分布

$$P(x) \equiv \frac{1}{\mathcal{U}_\kappa} \frac{p(x)}{\lambda u_\kappa\left(\frac{p(x)}{\alpha}\right)} \quad \underset{\kappa \to 0}{\longrightarrow} \quad p(x),$$

ここで、$\mathcal{U}_\kappa$ は規格化因子:

$$\mathcal{U}_\kappa \equiv \int dx \, \frac{p(x)}{\lambda u_\kappa\left(\frac{p(x)}{\alpha}\right)},$$

$A(x)$ の κ-escort 期待値 $\langle\!\langle A \rangle\!\rangle_\kappa$:

$$\langle\!\langle A \rangle\!\rangle_\kappa \equiv \int dx \, P(x) \, A(x).$$

$$\Rightarrow \quad \partial_i \gamma(\boldsymbol{\theta}) = \langle\!\langle f_i \rangle\!\rangle_\kappa \, ,$$

ℓ_θ の κ 拡張

$$\tilde{\ell}_\theta^{(\kappa)} \equiv \lambda \ln_\kappa \left(\frac{p_\theta(x)}{\alpha} \right) - \mathcal{I}_\kappa(\boldsymbol{\theta}),$$

which fulfills the relation:

$$\left\langle \partial_i \tilde{\ell}_\theta^{(\kappa)} \right\rangle = 0,$$

$\lim_{\kappa\to 0} \lambda = 1,\ \lim_{\kappa\to 0} \alpha = 1/e,\ \lim_{\kappa\to 0} \mathcal{I}_\kappa = 1$ なので、

$$\lim_{\kappa\to 0} \tilde{\ell}_\theta^{(\kappa)} = \ell_\theta = \ln p_\theta(x).$$

$$\eta_i \equiv \partial_i \Psi_\kappa(\boldsymbol{\theta}) = \langle f_i \rangle \,,$$

κ-変形計量:

$$g_{ij}^{(\kappa)} \equiv \partial_i \partial_j \, \Psi_\kappa(\boldsymbol{\theta}).$$

κ-変形 第 1 種 Christoffel 記号（e-接続):

$$\Gamma_{ij,k}^{(\kappa e)} \equiv \int dx \, \partial_i \partial_j \, \tilde{\ell}_\theta^{(\kappa)} \, \partial_k p(x;\boldsymbol{\theta}) = \left\langle \partial_i \partial_j \, \tilde{\ell}_\theta^{(\kappa)} \, \partial_k \, \ell_\theta \right\rangle ,$$

$$g_{ij}^{(\kappa)} = \mathcal{U}_\kappa \left\langle\!\left\langle (f_i - \langle\!\langle f_i \rangle\!\rangle_\kappa)(f_j - \langle\!\langle f_j \rangle\!\rangle_\kappa) \right\rangle\!\right\rangle_\kappa .$$

$$\eta_i \equiv \partial_i \Psi_\kappa(\boldsymbol{\theta}) = \langle f_i \rangle ,$$

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$$\Gamma_{ij,k}^{(\kappa e)} \equiv \int dx \, \partial_i \partial_j \, \tilde{\ell}_\theta^{(\kappa)} \, \partial_k p(x; \boldsymbol{\theta}) = \left\langle \partial_i \partial_j \, \tilde{\ell}_\theta^{(\kappa)} \, \partial_k \, \ell_\theta \right\rangle ,$$

$$g_{ij}^{(\kappa)} = \mathcal{U}_\kappa \, \langle\!\langle (f_i - \langle\!\langle f_i \rangle\!\rangle_\kappa)(f_j - \langle\!\langle f_j \rangle\!\rangle_\kappa) \rangle\!\rangle_\kappa .$$

$\Leftarrow$ κ-変形エスコート期待値に対する共分散

κ-generalized 大正準分布

$$p_\kappa^{\mathrm{G}}(x) = \alpha \ \exp_\kappa \left[\frac{1}{\lambda} \big(-\beta E_N(x) + \beta\mu N - \gamma(\beta,\mu) \big) \right].$$

$\Rightarrow$ κ-指数型分布族

$$\theta^1 = -\beta, \quad \theta^2 = \beta\,\mu, \quad f_1(x) = E_N(x), \quad f_2(x) = N,$$

$$\Psi_\kappa(\boldsymbol{\theta}) = \mathcal{I}_\kappa(\beta,\mu) + \gamma(\beta,\mu) = \Phi_\kappa^G(\beta,\mu),$$

$$\Psi_\kappa^*(\boldsymbol{\eta}) = -S_\kappa(\langle E_N \rangle, \langle N \rangle).$$

η-座標:

$$\eta_1^G = -\frac{\partial}{\partial\beta}\Phi_\kappa^G(\beta,\mu) = \langle E_n \rangle,$$

$$\eta_2^G = -\frac{\partial}{\partial(\beta\mu)}\Phi_\kappa^G(\beta,\mu) = \langle N \rangle,$$

κ-拡張版 揺動応答 (fluctuation-response) 関係

$$g^{(\kappa)\,\mathrm{G}} = \begin{pmatrix} -\frac{\partial}{\partial\beta}\langle E_N\rangle & -\frac{\partial}{\partial\beta}\langle N\rangle \\ \frac{\partial}{\partial(\beta\mu)}\langle E_N\rangle & \frac{\partial}{\partial(\beta\mu)}\langle N\rangle \end{pmatrix}$$
$$= \mathcal{U}_\kappa \begin{pmatrix} \langle\!\langle E_N^2\rangle\!\rangle_\kappa - \langle\!\langle E_N\rangle\!\rangle_\kappa^2 & \langle\!\langle E_N\, N\rangle\!\rangle_\kappa - \langle\!\langle E_N\rangle\!\rangle_\kappa \langle\!\langle N\rangle\!\rangle_\kappa \\ \langle\!\langle E_N\, N\rangle\!\rangle_\kappa - \langle\!\langle E_N\rangle\!\rangle_\kappa \langle\!\langle N\rangle\!\rangle_\kappa & \langle\!\langle N^2\rangle\!\rangle_\kappa - \langle\!\langle N\rangle\!\rangle_\kappa^2 \end{pmatrix}.$$

線形期待値 $\langle f_j\rangle$ に対する応答関数と κ-変形エスコート期待値に対する揺動との関係

J. Naudts: *J. Inequal. Pure and Appl. Math.*, **5**, Art. 102 (2004). .

まとめ

κ-変形指数分布に基づいた情報幾何構造を構成:

- 双対平坦: 双対ポテンシャル： $\Psi_\kappa = \mathcal{I}_\kappa + \gamma$ and $\Psi^*_\kappa = -S_\kappa$.
- κ-変形指数分布に基づく 揺動-応答関係の拡張

詳細は:

T.W and A.M. Scarfone:
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